Asymptotic Analysis of High-Contrast Phononic Crystals and a Criterion for the Band-Gap Opening[*]

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Abstract

We investigate the band-gap structure of the frequency spectrum for elastic waves in a high-contrast, two-component periodic elastic medium. We consider two-dimensional phononic crystals consisting of a background medium which is perforated by an array of holes periodic along each of the two orthogonal coordinate axes. In this paper we establish a full asymptotic formula for dispersion relations of phononic band structures as the contrast of the shear modulus and that of the density become large. The main ingredients are integral equation formulations of the solutions to the harmonic oscillatory linear elastic equation and several theorems concerning the characteristic values of meromorphic operator-valued functions in the complex plane such as Generalized Rouché's theorem. We establish a connection between the band structures and the Dirichlet eigenvalue problem on the elementary hole. We also provide a criterion for exhibiting gaps in the band structure which shows that smaller the density of the matrix is, wider the band-gap is, provided that the criterion is fulfilled. This phenomenon was reported by Economou and Sigalas in [14] who observed that periodic elastic composites whose matrix has lower density and higher shear modulus compared to those of inclusions yield better open gaps. Our analysis in this paper agrees with this experimental finding.

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1 Introduction

In the past decade there has been a steady growth of interest in the motion of elastic waves through inhomogeneous materials. The primary motive for these investigations has been the design of the so-called phononic band gap materials or phononic crystals. The most recent research in this field has focused on theoretical and experimental demonstration of band gaps in two-dimensional and three-dimensional structures constructed of high-contrast elastic materials arranged in a periodic array. This type of structure prevents elastic waves in certain frequency ranges from propagating and could be used to generate frequency filters with control of pass or stop bands, as beam splitters, as sound or vibration protection devices, or as elastic waveguides. See, for example, [32, 12, 25, 30].

The interest in phononic crystals has been renewed by the work in optics on photonic band gap materials. Since the seminal paper by Yablonovitch [33], significant progress has been made in microstructuring a dielectric or magnetic material on the scale of the optical wavelength so that a range of frequencies for which incident electromagnetic waves are unable to propagate through the designed crystal exists. See [21, 15, 16, 23]. See also [34] for extensive list of references on photonic crystals.

To formulate the investigation of this paper in mathematical terms, let D be a connected domain with the Lipschitz boundary lying inside the open square $]0, 1[^2$. An important example of phononic crystals consists of a background elastic medium of constant Lamé parameters λ and μ which is perforated by an array of arbitrary-shaped inclusions $\Omega = \cup_{n \in \mathbb{Z}^2}(D + n)$ periodic along each of the two orthogonal coordinate axes in the plane. These inclusions have Lamé constants $\tilde{\lambda}$, $\tilde{\mu}$. The shear modulus μ of the background medium is assumed to be larger than that of the inclusion $\tilde{\mu}$. Then we investigate the spectrum of the self-adjoint operator defined by

$$\mathbf{u} \mapsto -\nabla \cdot (C\nabla \mathbf{u}) = -\sum_{j,k,l=1}^{2} \frac{\partial}{\partial x_j}\left(C_{ijkl}\frac{\partial u_k}{\partial x_l}\right), \tag{1.1}$$

which is densely defined on $L^2(\mathbb{R}^2)^2$. Here the elasticity tensor C is given by

$$C_{ijkl} := \left(\lambda\,\chi(\mathbb{R}^2 \setminus \overline{\Omega}) + \tilde{\lambda}\,\chi(\Omega)\right)\delta_{ij}\delta_{kl} + \left(\mu\,\chi(\mathbb{R}^2 \setminus \overline{\Omega}) + \tilde{\mu}\,\chi(\Omega)\right)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{1.2}$$

where $\chi(\Omega)$ is the indicator function of Ω.

In this paper we adopt this specific two-dimensional model to understand the relationship between the contrast of the shear modulus and the band gap structure of the phononic crystal. We will also consider the case of two materials with different densities in order to investigate the relation between the density contrast and the band gap.

By Floquet theory [22], the spectrum of the Lamé system with periodic coefficients is represented as a union of bands, called phononic band structure. Carrying out a band structure calculation for a given phononic crystal involves a family of eigenvalue problems, as the quasi-momentum is varied over the first Brillouin zone. The problem of finding the spectrum of (1.1) is reduced to a family of eigenvalue problems with quasi-periodicity condition, *i.e.*,

$$\nabla \cdot (C\nabla\mathbf{u}) + \omega^2\mathbf{u} = 0 \text{ in } \mathbb{R}^2, \tag{1.3}$$

with the periodicity condition

$$\mathbf{u}(x+n) = e^{i\alpha\cdot n}\mathbf{u}(x) \quad \text{for every } n \in \mathbb{Z}^2. \tag{1.4}$$

Here the quasi-momentum α varies over the Brillouin zone $[0, 2\pi[^2$. Each of these operators has compact resolvent so that its spectrum consists of discrete eigenvalues of finite multiplicity. We show that these eigenvalues are the characteristic values of meromorphic operator-valued functions that are of Fredholm type with index zero. This yields a new and natural approach to the computation of the band gap phononic structure which is based on a combination of boundary element methods and Muller's method [31] for finding complex roots of scalar equations. Efficiency of a similar approach for computing photonic band gaps has been demonstrated in [11, 12]. We then proceed from the generalized Rouché's theorem to construct their complete asymptotic expressions as the Lamé parameter μ of the background goes to infinity. For $\alpha \neq 0$, we prove that the discrete spectrum of (1.3) accumulates near the Dirichlet eigenvalues of Lamé system in D as μ goes to infinity. We then obtain a full asymptotic formula for the eigenvalues with the leading order term of order μ^{-1} calculated explicitly. For the periodic case $\alpha = 0$, we establish a formula for asymptotic behavior of eigenvalues, but their limiting set is generically different from that for $\alpha \neq 0$. We also consider the case when $|\alpha|$ is of order $1/\sqrt{\mu}$ and derive an asymptotic expansion for the eigenvalues in this case as well. Not surprisingly, this formula tends continuously to the previous ones as $\alpha\sqrt{\mu}$ goes to zero or to infinity. We finally provide a criterion for exhibiting gaps in the band structure. As has been said, the existence of those spectral gaps implies that the elastic waves in those frequency ranges are prohibited from travelling through the elastic body. Our criterion shows that smaller the density of the matrix is, wider the band-gap is, provided that the criterion is fulfilled. This phenomenon was reported by Economou and Sigalas in [14] who observed that periodic elastic composites whose matrix has lower density and higher shear modulus compared to those of inclusions yield better open gaps.

Similar results for the photonic crystals were obtained by Hempel and Lienau in [20], where they dealt with conductivity equation with high contrast in two phase composites. See also Friedlander [17]. Another related work is [7] which concerns the photonic band gap using the same method as the one in this work. A justification of the existence of elastic band gaps in periodic composite materials with strong heterogeneities has been recently provided by Ávila et al. in [9] by extending Bouchitté and Felbacq scalar homogenization approach [10] to the elasticity problem. We also mention works by Movchan and his collaborators [27, 29, 28, 26]. To the best of our knowledge, our result on the gap opening is achieved by

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using a method significantly different from those in the literature. All the other asymptotic results of this paper are new and have never been established elsewhere.

The main ingredients in deriving the results of this paper are the boundary integral equations and the theory of meromorphic operator-valued functions. Using integral representations of the solutions to the harmonic oscillatory linear elastic equation, we reduce this problem to the study of characteristic values of integral operators in the complex planes. Generalized Rouché's theorem and other techniques from the theory of meromorphic operator-valued functions are combined with careful asymptotic expansions of integral kernels to obtain full asymptotic expansions for eigenvalues. This method was first used in [8], and then successfully applied to obtain an asymptotic formula for the eigenvalues of Laplacian under singular perturbations [6] and high contrast asymptotics for the photonic crystals [7]. See also [4].

Results of this paper could be used to design an optimization tool based on layer potential techniques for the systematic design of the band-gap elastic materials and structures. Since the limiting situation reduces to easy-to-calculate spectra, the idea would be to start with these spectra (as initial guess) and then compute the gradient of some target functional using our asymptotic expansions with respect to the contrast. Moreover, in order to optimize the position and width of these gaps, we only need to optimize the shape of the inclusion considering the (more simpler) limiting situation.

The paper is organized as follows. In section 2, we recall relevant definitions and state the generalized Rouché's theorem. Then we introduce the single and the double layer potentials for the harmonic oscillatory linear elastic equation and cover well-known results for the quasi-periodic case as well. In section 3, we state and prove our main results. Section 4 is devoted to the derivation of a criterion for gap opening in the spectrum of the operator given by (1.1) as $\mu \to +\infty$. In section 5, we derive a similar criterion when the contrast of the density is high.

2   Preliminaries

The first subsection of this section covers relevant parts of the theory of meromorphic operator-valued functions. We state the generalized Rouché's theorem. In the second subsection we review some well-known results on the solvability and theory of layer potentials for the harmonic oscillatory linear elastic equation, which we shall use frequently throughout this paper. The third subsection is devoted to the radiation condition for elastic wave propagation. In the fourth subsection we collect some notation and well-known results regarding quasi-periodic layer potentials for the harmonic oscillatory linear elastic equation.

2.1   The generalized Rouché's theorem

In this work the approach we develop is a boundary integral technique with rigorous justification based on the generalized Rouché's theorem. For readers' convenience we recall this the main results of Gohberg and Sigal in [18]. We begin by collecting some definitions.

Let $\mathcal{G}$ and $\mathcal{H}$ be two Banach spaces and let $\mathcal{L}(\mathcal{G}, \mathcal{H})$ be the set of all bounded operators from $\mathcal{G}$ to $\mathcal{H}$. $A \in \mathcal{L}(\mathcal{G}, \mathcal{H})$ is called a *Fredholm operator* if A is closed and Ker A and Coker A are finite dimensional. If A is Fredholm, the index of A is defined to be

$$\text{Ind } A = \dim \text{Ker } A - \dim \text{Coker } A. \tag{2.1}$$

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If A is Fredholm, then for any compact operator $K \in \mathcal{L}(\mathcal{G}, \mathcal{H})$, $A + K$ is also Fredholm with index Ind A.

Let U be an open set in $\mathbb{C}$. Suppose that $\omega \mapsto A(\omega)$ is an operator-valued function from U into $\mathcal{L}(\mathcal{G}, \mathcal{H})$. We call $\omega_0 \in U$ a *characteristic value* of $A(\omega)$ if

- $A(\omega)$ is holomorphic in some neighborhood of ω_0, except possibly for ω_0;

- There exists a function $\phi(\omega)$ from a neighborhood of ω_0 to $\mathcal{G}$, holomorphic and nonzero at ω_0, such that $A(\omega)\phi(\omega)$ is holomorphic at ω_0 and $A(\omega_0)\phi(\omega_0) = 0$.

The function $\phi(\omega)$ in the above definition is called a *root function* of $A(\omega)$ associated to ω_0 and $\phi(\omega_0)$ is called an *eigenvector*. The closure of the space of eigenvectors corresponding to ω_0 is denoted by Ker $A(\omega_0)$.

Let ϕ_0 be an eigenvector corresponding to ω_0. Let $V(\omega_0)$ be a complex neighborhood of ω_0. The *rank* of ϕ_0 is the largest integer m such that there exist $\phi : V(\omega_0) \to \mathcal{G}$ and $\psi : V(\omega_0) \to \mathcal{H}$ holomorphic satisfying

$$A(\omega)\phi(\omega) = (\omega - \omega_0)^m \psi(\omega), \quad \phi(\omega_0) = \phi_0, \quad \psi(\omega_0) \neq 0. \tag{2.2}$$

Suppose that $A(\omega)$ admits the Laurent expansion in ω_0 such as

$$A(\omega) = \sum_{j \geq -s} (\omega - \omega_0)^j A_j \tag{2.3}$$

for some non-negative integer s, where A_j are operators in $\mathcal{L}(\mathcal{G}, \mathcal{H})$ independent of ω. If the operators A_j, $j = -s, \cdots, -1$, are finite dimensional, then $A(\omega)$ is called *finitely meromorphic* at ω_0. If the operator A_0 is a Fredholm operator, $A(\omega)$ is said to be of *Fredholm type* at ω_0.

If $A(\omega)$ is holomorphic at ω_0 and $A(\omega_0)$ is invertible, then ω_0 is called a *regular point* of $A(\omega)$. A point ω_0 is called a *normal point* of $A(\omega)$ if $A(\omega)$ is finitely meromorphic and of Fredholm type at ω_0 and there exists some neighborhood $V(\omega_0)$ of ω_0 in which all the points except ω_0 are regular points of $A(\omega)$. The following holds.

Lemma 2.1 *Every normal point ω_0 of $A(\omega)$ is a normal point of $A^{-1}(\omega)$.*

Suppose that ω_0 is a normal point of $A(\omega)$. Then $\dim \operatorname{Ker} A(\omega_0) < +\infty$ and the ranks of all vectors in Ker $A(\omega_0)$ are finite. A system of eigenvectors ϕ^j, $j = 1, \cdots, n$, is called a *canonical system of eigenvectors* of $A(\omega)$ associated to ω_0 if $\{\phi^1, \ldots, \phi^n\}$ is a basis of Ker $A(\omega_0)$ and the rank of ϕ^j is the maximum of the ranks of all eigenvectors in some direct complement in Ker $A(\omega_0)$ of the linear span of the vectors $\phi^1, \cdots, \phi^{j-1}$. Then we define the *null multiplicity* of the characteristic value ω_0 of $A(\omega)$ by

$$N(A(\omega_0)) = \sum_{i=1}^{n} \operatorname{rank}(\phi^i). \tag{2.4}$$

Similarly, we can also define $N(A^{-1}(\omega_0))$. Then the number

$$M(A(\omega_0)) = N(A(\omega_0)) - N(A^{-1}(\omega_0)) \tag{2.5}$$

is called the *multiplicity* of the characteristic value ω_0 of $A(\omega)$.

For an open subset V of U, an operator-valued function $A(\omega)$ is called *normal* with respect to ∂V if $A(\omega)$ is invertible and continuous at ∂V and all the points of V are regular, except for a finite number of normal points of $A(\omega)$.

Suppose that $A(\omega)$ is normal with respect to ∂V and let ω_i, $i = 1, \cdots, \sigma$, be a finite number of normal points of $A(\omega)$ in V. Then we define

$$\mathcal{M}(A(\omega); \partial V) = \sum_{i=1}^{\sigma} M(A(\omega_i)). \tag{2.6}$$

The generalization of Rouché's theorem was obtained in [18].

Theorem 2.2 *Let $A(\omega)$ be an operator-valued function which is normal with respect to ∂V. If an operator-valued function $S(\omega)$ which is finitely meromorphic in V and continuous on ∂V satisfies the condition*

$$\|A^{-1}(\omega)S(\omega)\|_{\mathcal{L}(\mathcal{G},\mathcal{G})} < 1, \quad \omega \in \partial V, \tag{2.7}$$

then $A(\omega) + S(\omega)$ is also normal with respect to ∂V and

$$\mathcal{M}(A(\omega); \partial V) = \mathcal{M}(A(\omega) + S(\omega); \partial V). \tag{2.8}$$

The generalization of Steinberg theorem is given by the following.

Theorem 2.3 *Suppose that $A(\omega)$ is an operator-valued function which is finitely meromorphic and of Fredholm type in V. If $A(\omega)$ is invertible at one point of V, then $A(\omega)$ has a bounded inverse for all $\omega \in V$, except possibly for certain isolated points.*

The operator generalization of the residue theorem is as follows.

Theorem 2.4 *Suppose that $A(\omega)$ is an operator-valued function which is normal with respect to ∂V. Let $f(\omega)$ be a scalar function which is holomorphic in V and continuous on ∂V. Then we have*

$$\frac{1}{2\pi\sqrt{-1}} \operatorname{tr} \int_{\partial V} f(\omega) A^{-1}(\omega) \frac{d}{d\omega} A(\omega) d\omega = \sum_{j=1}^{\sigma} M(A(\omega_j)) f(\omega_j), \tag{2.9}$$

where ω_j, $j = 1, \cdots, \sigma$, are all the points in V which are either poles or characteristic values of $A(\omega)$.

Here tr denotes the trace of operator which is the sum of all its nonzero eigenvalues. The following property of the trace is of use to us:

$$\operatorname{tr} \int_{\partial V(\omega_0)} A(\omega) B(\omega) d\omega = \operatorname{tr} \int_{\partial V(\omega_0)} B(\omega) A(\omega) d\omega, \tag{2.10}$$

where $A(\omega)$ and $B(\omega)$ are operator-valued functions which are finitely meromorphic in $V(\omega_0)$, which contains no poles of $A(\omega)$ and $B(\omega)$ other than ω_0.

2.2 Green's tensor and layer potentials

Let Ω be a bounded domain in $\mathbb{R}^2$ with a connected Lipschitz boundary $\partial\Omega$. Let the space $H^1(\partial\Omega)$ be the set of functions $f \in L^2(\partial\Omega)$ such that $\partial f/\partial T \in L^2(\partial\Omega)$, where $\partial/\partial T$ denotes the tangential derivative on $\partial\Omega$. We use $H^s(\Omega)$ as a notation for the standard Sobolev space of order s.

Let λ, μ be the Lamé constants for Ω satisfying

$$\mu > 0 \quad \text{and} \quad 3\lambda + 2\mu > 0. \tag{2.11}$$

The elastostatic system corresponding to the Lamé constants λ, μ is defined by

$$\mathcal{L}^{\lambda,\mu}\mathbf{u} = \mu\triangle\mathbf{u} + (\lambda + \mu)\nabla\nabla \cdot \mathbf{u}, \tag{2.12}$$

and the corresponding conormal derivative $\partial\mathbf{u}/\partial\nu$ on $\partial\Omega$ is defined to be

$$\frac{\partial\mathbf{u}}{\partial\nu} = \lambda(\nabla \cdot \mathbf{u})N + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^t)N, \tag{2.13}$$

where N is the outward unit normal to $\partial\Omega$ and the superscript t denotes the transpose of a matrix.

The fundamental solution $\mathbf{\Gamma}^\omega = (\Gamma_{ij}^\omega)_{i,j=1}^2$ to the operator $\mathcal{L}^{\lambda,\mu} + \omega^2$, $\omega \neq 0$, is given by

$$\Gamma_{ij}^\omega(x) = -\frac{\sqrt{-1}}{4\mu}\delta_{ij}H_0^{(1)}\left(\frac{\omega|x|}{c_T}\right) + \frac{\sqrt{-1}}{4\omega^2}\partial_i\partial_j\left(H_0^{(1)}\left(\frac{\omega|x|}{c_L}\right) - H_0^{(1)}\left(\frac{\omega|x|}{c_T}\right)\right), \tag{2.14}$$

where δ_{ij} is the Kronecker's delta, ∂_j denotes $\partial/\partial x_j$, $c_T = \sqrt{\mu}$, $c_L = \sqrt{\lambda + 2\mu}$, and $H_0^{(1)}$ is the Hankel function of the first kind and of order 0. See [1] and [24, Chap. 2]. For $\omega = 0$, we set $\mathbf{\Gamma}^0$ to be the Kelvin matrix of fundamental solutions to the Lamé system, *i.e.*,

$$\Gamma_{ij}^0(x) = \frac{\gamma_1}{2\pi}\delta_{ij}\ln|x| - \frac{\gamma_2}{2\pi}\frac{x_ix_j}{|x|^2}, \tag{2.15}$$

where

$$\gamma_1 = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{2\mu + \lambda}\right) \quad \text{and} \quad \gamma_2 = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{2\mu + \lambda}\right). \tag{2.16}$$

The single and double layer potentials of the density function $\varphi \in L^2(\partial\Omega)^2$ associated with the Lamé parameters (λ, μ) are defined by

$$\mathcal{S}^\omega\varphi(x) = \int_{\partial\Omega}\mathbf{\Gamma}^\omega(x - y)\varphi(y)d\sigma(y), \quad x \in \mathbb{R}^2, \tag{2.17}$$

$$\mathcal{D}^\omega\varphi(x) = \int_{\partial\Omega}\frac{\partial}{\partial\nu_y}\mathbf{\Gamma}^\omega(x - y)\varphi(y)d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial\Omega \tag{2.18}$$

where $\frac{\partial}{\partial\nu_y}$ denotes the conormal derivative with respect to the y-variables.

The following formulae give the jump relations obeyed by the double layer potential and by the conormal derivative of the single layer potential:

$$\frac{\partial(\mathcal{S}^\omega\varphi)}{\partial\nu}\Big|_{\pm}(x) = \left(\pm\frac{1}{2}I + (\mathcal{K}^\omega)^*\right)\varphi(x), \quad \text{a.e. } x \in \partial\Omega, \tag{2.19}$$

$$(\mathcal{D}^\omega\varphi)\Big|_{\pm}(x) = \left(\mp\frac{1}{2}I + \mathcal{K}^\omega\right)\varphi(x), \quad \text{a.e. } x \in \partial\Omega, \tag{2.20}$$

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where the subscripts $\pm$ indicate the limit from outside and inside Ω, respectively, the singular integral operator $\mathcal{K}^\omega$ is defined by

$$\mathcal{K}^\omega(x) = \text{p.v.} \int_{\partial\Omega} \frac{\partial \mathbf{\Gamma}^\omega(x-y)}{\partial \nu_y} \varphi(y) d\sigma(y), \tag{2.21}$$

and $(\mathcal{K}^\omega)^*$ is its L^2-adjoint, that is,

$$(\mathcal{K}^\omega)^*(x) = \text{p.v.} \int_{\partial\Omega} \frac{\partial \mathbf{\Gamma}^\omega(x-y)}{\partial \nu_x} \varphi(y) d\sigma(y). \tag{2.22}$$

Here p.v. means the Cauchy principal value. See [24, 13].

It is proved in [2, 13] that $\mathcal{S}^\omega$ is Fredholm as an operator from $L^2(\partial\Omega)^2$ into $H^1(\partial\Omega)^2$ and has index zero. It is also proved that the operators $\pm\frac{1}{2}I + \mathcal{K}^\omega$ and $\pm\frac{1}{2}I + (\mathcal{K}^\omega)^*$ are also Fredholm on $L^2(\partial\Omega)^2$ with index 0. We should emphasize that $\mathcal{K}^\omega$ is not compact, even on bounded $\mathcal{C}^\infty$-domains [13].

Let Ψ be the vector space of all linear solutions to the equation $\mathcal{L}^{\lambda,\mu}\mathbf{u} = 0$ and $\dfrac{\partial \mathbf{u}}{\partial \nu} = 0$ on $\partial\Omega$, or alternatively,

$$\Psi = \big\{ \psi : \partial_i \psi_j + \partial_j \psi_i = 0, \ 1 \le i,j \le 2 \big\}.$$

Here ψ_i for $i = 1,2$, denote the components of ψ. Define

$$L^2_\Psi(\partial\Omega) = \Big\{ \mathbf{f} \in L^2(\partial\Omega)^2 : \int_{\partial\Omega} \mathbf{f} \cdot \psi d\sigma = 0 \text{ for all } \psi \in \Psi \Big\},$$

a subspace of codimension 3 in $L^2(\partial\Omega)^2$. In particular, since Ψ contains constant functions, we get

$$\int_{\partial\Omega} \mathbf{f} d\sigma = 0 \quad \text{for any } \mathbf{f} \in L^2_\Psi(\partial\Omega).$$

We recall Green's formulae for the Lamé system, which can be obtained by integration by parts. The first formula is

$$\int_{\partial\Omega} \mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial \nu} d\sigma = \int_{\Omega} \mathbf{u} \cdot \mathcal{L}^{\lambda,\mu}\mathbf{v} + \mathbf{E}(\mathbf{u}, \mathbf{v}), \tag{2.23}$$

where $\mathbf{u} \in H^1(\Omega)^2$, $\mathbf{v} \in H^{\frac{3}{2}}(\Omega)^2$, and

$$\mathbf{E}(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \lambda(\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v}) + \frac{\mu}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t) \cdot (\nabla\mathbf{v} + \nabla\mathbf{v}^t). \tag{2.24}$$

Formula (2.23) yields Green's second formula

$$\int_{\partial\Omega} \Big(\mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial \nu} - \mathbf{v} \cdot \frac{\partial \mathbf{u}}{\partial \nu} \Big) = \int_{\Omega} (\mathbf{u} \cdot \mathcal{L}^{\lambda,\mu}\mathbf{v} - \mathbf{v} \cdot \mathcal{L}^{\lambda,\mu}\mathbf{u}), \tag{2.25}$$

where $\mathbf{u}$, $\mathbf{v} \in H^{\frac{3}{2}}(\Omega)^2$.

Formula (2.25) shows that if $\mathbf{u} \in H^{\frac{3}{2}}(\Omega)^2$ satisfies $\mathcal{L}^{\lambda,\mu}\mathbf{u} = 0$ in Ω, then $\dfrac{\partial \mathbf{u}}{\partial \nu}\Big|_{\partial\Omega} \in L^2_\Psi(\partial\Omega)$.

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2.3 Radiation conditions and uniqueness

Let us now formulate the *radiation condition* for the elastic waves when $\operatorname{Im} \omega \geq 0$ and $\omega \neq 0$. See [1, 2, 24]. Any solution $\mathbf{u}$ to $(\mathcal{L}^{\lambda,\mu} + \omega^2)\mathbf{u} = 0$ admits the decomposition,

$$\mathbf{u} = \mathbf{u}^{(p)} + \mathbf{u}^{(s)}, \tag{2.26}$$

where $\mathbf{u}^{(p)}$ and $\mathbf{u}^{(s)}$ are given by

$$\mathbf{u}^{(p)} = (k_T^2 - k_L^2)^{-1}(\triangle + k_T^2)\mathbf{u}, \tag{2.27}$$

$$\mathbf{u}^{(s)} = (k_L^2 - k_T^2)^{-1}(\triangle + k_L^2)\mathbf{u}, \tag{2.28}$$

with

$$k_T = \frac{\omega}{c_T} = \frac{\omega}{\sqrt{\mu}} \quad \text{and} \quad k_L = \frac{\omega}{c_L} = \frac{\omega}{\sqrt{\lambda + 2\mu}}. \tag{2.29}$$

Then $\mathbf{u}^{(p)}$ and $\mathbf{u}^{(s)}$ satisfy the equations

$$\begin{cases} (\triangle + k_T^2)\mathbf{u}^{(p)} = 0, & \operatorname{rot} \mathbf{u}^{(p)} = 0, \\ (\triangle + k_L^2)\mathbf{u}^{(s)} = 0, & \nabla \cdot \mathbf{u}^{(s)} = 0. \end{cases} \tag{2.30}$$

We impose on $\mathbf{u}^{(p)}$ and $\mathbf{u}^{(s)}$ the Sommerfeld radiation conditions for the solutions of the corresponding Helmholtz equations by requiring that

$$\begin{cases} \partial_r \mathbf{u}^{(p)}(x) - \sqrt{-1} k_T \mathbf{u}^{(p)}(x) = o(r^{-\frac{1}{2}}), \\ \partial_r \mathbf{u}^{(s)}(x) - \sqrt{-1} k_L \mathbf{u}^{(s)}(x) = o(r^{-\frac{1}{2}}), \end{cases} \quad \text{as } r = |x| \to +\infty. \tag{2.31}$$

We say that $\mathbf{u}$ satisfies the radiation condition if it allows the decomposition (2.26) with $\mathbf{u}^{(p)}$ and $\mathbf{u}^{(s)}$ satisfying (2.30) and (2.31). By a straightforward calculation one can see that (non-periodic) single and double layer potentials, $\mathcal{S}^\omega \phi$ and $\mathcal{D}^\omega \phi$, satisfy the radiation condition. See [2, 24].

We recall the following uniqueness results for the exterior problem [24].

Lemma 2.5 *Let $\mathbf{u}$ be a solution to $(\mathcal{L}^{\lambda,\mu} + \omega^2)\mathbf{u} = 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$ satisfying the radiation condition. If either $\mathbf{u} = 0$ or $\frac{\partial \mathbf{u}}{\partial \nu} = 0$ on $\partial \Omega$, then $\mathbf{u}$ is identically zero in $\mathbb{R}^2 \setminus \overline{\Omega}$.*

Let U be a bounded and connected open subset of $\mathbb{R}^2$ with the Lipschitz boundary such that $\overline{\Omega} \subset U$. Let $D_1 = U \setminus \overline{\Omega}$ and $D_2 = (\mathbb{R}^2 \setminus \overline{\Omega}) \setminus \overline{D}_1$. Let $\mathcal{L}^{\tilde{\lambda},\tilde{\mu}}$ be the operator defined by (2.12) and $\frac{\partial}{\partial \tilde{\nu}}$ be the corresponding conormal derivative. Consider the following two-phase transmission problem:

$$\begin{cases} \mathcal{L}^{\tilde{\lambda},\tilde{\mu}} \mathbf{u} + \omega^2 \mathbf{u} = 0, & \text{in } D_1, \\ \mathcal{L}^{\lambda,\mu} \mathbf{u} + \omega^2 \mathbf{u} = 0, & \text{in } D_2, \\ \mathbf{u}\big|_+ - \mathbf{u}\big|_- = 0, & \text{on } \partial D_1 \cap \partial D_2, \\ \dfrac{\partial \mathbf{u}}{\partial \nu}\Big|_+ - \dfrac{\partial \mathbf{u}}{\partial \tilde{\nu}}\Big|_- = 0, & \text{on } \partial D_1 \cap \partial D_2, \end{cases} \tag{2.32}$$

with the radiation condition. The following uniqueness result holds.

Lemma 2.6 *Let $\mathbf{u}$ be a solution to (2.32) in $\mathbb{R}^2 \setminus \overline{\Omega}$. If either $\mathbf{u} = 0$ or $\frac{\partial \mathbf{u}}{\partial \nu} = 0$ on $\partial \Omega$, $\mathbf{u}$ is identically zero in $\mathbb{R}^2 \setminus \overline{\Omega}$.*

We note that the above two lemmas hold even when Ω is an empty set.

2.4   Quasi-periodic Green's function

In this section we collect some notation and well-known results regarding quasi-periodic layer potentials for the Lamé system in $\mathbb{R}^2$. We refer to [27, 29, 19] for the details.

We assume that the unit cell $Y = [0, 1[^2$ is the periodic cell and the quasi-momentum variable, denoted by α, ranges over the Brillouin zone $B = [0, 2\pi[^2$. We introduce the two-dimensional quasi-periodic Green's function $\mathbf{G}^{\alpha,\omega}$ for $\omega \neq 0$, which satisfies

$$(\mathcal{L}^{\lambda,\mu} + \omega^2)\mathbf{G}^{\alpha,\omega}(x, y) = \sum_{n \in \mathbb{Z}^2} \delta(x - y - n)e^{\sqrt{-1}n \cdot \alpha}I. \tag{2.33}$$

Here we assume that $k_T, k_L \neq |2\pi n + \alpha|$ for all $n \in \mathbb{Z}^2$ where k_T and k_L are given by (2.29).

A function $\mathbf{u}$ is said to be quasi-periodic or α-quasi-periodic if $e^{-\sqrt{-1}\alpha \cdot x}\mathbf{u}$ is periodic. Using Poisson's summation formula, we have

$$\sum_{n \in \mathbb{Z}^2} \delta(x - y - n)e^{\sqrt{-1}n \cdot \alpha}I = \sum_{n \in \mathbb{Z}^2} e^{\sqrt{-1}(2\pi n + \alpha)\cdot(x - y)}I.$$

We plug this equation into (2.33) and then take the Fourier transform of both sides of (2.33) to obtain

$$\widehat{G}_{ij}^{\alpha,\omega}(\xi, y) = (2\pi)^2 \left\{ \frac{\delta_{ij}}{c_T^2} \frac{1}{k_T^2 - \xi^2} + \frac{\xi_i \xi_j}{\omega^2}\left(\frac{1}{k_L^2 - \xi^2} - \frac{1}{k_T^2 - \xi^2}\right)\right\}$$
$$\times \sum_{n \in \mathbb{Z}^2} e^{-\sqrt{-1}(2\pi n + \alpha)\cdot y}\delta(\xi + 2\pi n + \alpha),$$

where $\xi^2 = \xi \cdot \xi$ and $\widehat{}$ denotes the Fourier transform. Then taking the inverse Fourier transform, we can see that the quasi-periodic Green's function $\mathbf{G}^{\alpha,\omega} = (G_{ij}^{\alpha,\omega})$ can be represented as a sum of augmented plane waves over the reciprocal lattice:

$$G_{ij}^{\alpha,\omega}(x, y) = \frac{\delta_{ij}}{c_T^2} \sum_{n \in \mathbb{Z}^2} \frac{e^{\sqrt{-1}(2\pi n + \alpha)\cdot(x - y)}}{k_T^2 - |2\pi n + \alpha|^2}$$
$$+ \frac{k_T^2 - k_L^2}{\omega^2} \sum_{n \in \mathbb{Z}^2} \frac{e^{\sqrt{-1}(2\pi n + \alpha)\cdot(x - y)}(2\pi n + \alpha)_i (2\pi n + \alpha)_j}{(k_L^2 - |2\pi n + \alpha|^2)(k_T^2 - |2\pi n + \alpha|^2)}. \tag{2.34}$$

Moreover, it can also be easily shown that

$$\mathbf{G}^{\alpha,\omega}(x, y) = \sum_{n \in \mathbb{Z}^2} \mathbf{\Gamma}^{\omega}(x - n - y)e^{\sqrt{-1}n \cdot \alpha}, \tag{2.35}$$

where $\mathbf{\Gamma}^{\omega}$ is the Green's matrix defined by (2.14).

When $\omega = 0$, we define $\mathbf{G}^{\alpha,0}$ by

$$G_{ij}^{\alpha,0}(x, y) := \frac{1}{\mu}\sum_{n \in \mathbb{Z}^2} e^{\sqrt{-1}(2\pi n + \alpha)\cdot(x - y)}\left(\frac{-\delta_{ij}}{|2\pi n + \alpha|^2} + \frac{\lambda + \mu}{\lambda + 2\mu}\frac{(2\pi n + \alpha)_i (2\pi n + \alpha)_j}{|2\pi n + \alpha|^4}\right) \tag{2.36}$$

if $\alpha \neq 0$, while if $\alpha = 0$, it is defined by

$$G_{ij}^{0,0}(x, y) := \frac{1}{\mu}\sum_{n \neq (0,0)} e^{\sqrt{-1}2\pi n \cdot(x - y)}\left(\frac{-\delta_{ij}}{|2\pi n|^2} + \frac{\lambda + \mu}{\lambda + 2\mu}\frac{4\pi^2 n_i n_j}{|2\pi n|^4}\right) \tag{2.37}$$

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Then $\mathbf{G}^{\alpha,0}$ is quasi-periodic and satisfies

$$\mathcal{L}^{\lambda,\mu}\mathbf{G}^{\alpha,0}(x,y) = \sum_{n\in\mathbb{Z}^2} \delta(x-y-n)I \quad \text{if } \alpha \neq 0, \tag{2.38}$$

$$\mathcal{L}^{\lambda,\mu}\mathbf{G}^{0,0}(x,y) = \sum_{n\in\mathbb{Z}^2} \delta(x-y-n)I - I. \tag{2.39}$$

See [5, 3] for the proof.

Let D be a bounded domain in $\mathbb{R}^2$ with a connected Lipschitz boundary ∂D. Let $\mathcal{S}^{\alpha,\omega}$ and $\mathcal{D}^{\alpha,\omega}$ be the quasi-periodic single and double layer potentials associated with $\mathbf{G}^{\alpha,\omega}$, that is, for a given density $\varphi \in L^2(\partial D)^2$,

$$\mathcal{S}^{\alpha,\omega}\varphi(x) = \int_{\partial D} \mathbf{G}^{\alpha,\omega}(x,y)\varphi(y)\,d\sigma(y), \quad x \in \mathbb{R}^2,$$

$$\mathcal{D}^{\alpha,\omega}\varphi(x) = \int_{\partial D} \frac{\partial \mathbf{G}^{\alpha,\omega}(x,y)}{\partial \nu_y}\varphi(y)\,d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial D,$$

where $\frac{\partial}{\partial \nu_y}$ denotes the conormal derivative with respect to y. Then, $\mathcal{S}^{\alpha,\omega}\varphi$ and $\mathcal{D}^{\alpha,\omega}\varphi$ are solutions to

$$(\mathcal{L}^{\lambda,\mu} + \omega^2)\mathbf{u} = 0$$

in D and $Y \setminus \overline{D}$ and they are α-quasi-periodic.

The next formulae give the jump relations obeyed by the double layer potential and by the normal derivative of the single layer potential on general Lipschitz domains:

$$\left.\frac{\partial(\mathcal{S}^{\alpha,\omega}\varphi)}{\partial \nu}\right|_{\pm}(x) = \left(\pm\frac{1}{2}I + (\mathcal{K}^{\alpha,\omega})^*\right)\varphi(x) \quad \text{a.e. } x \in \partial D, \tag{2.40}$$

$$\left.(\mathcal{D}^{\alpha,\omega}\varphi)\right|_{\pm}(x) = \left(\mp\frac{1}{2}I + \mathcal{K}^{-\alpha,\omega}\right)\varphi(x) \quad \text{a.e. } x \in \partial D, \tag{2.41}$$

for $\varphi \in L^2(\partial D)^2$, where $\mathcal{K}^{\alpha,\omega}$ is the operator on $L^2(\partial D)^2$ defined by

$$\mathcal{K}^{\alpha,\omega}\varphi(x) = \text{p.v.} \int_{\partial D} \frac{\partial \mathbf{G}^{-\alpha,\omega}(x,y)}{\partial \nu_y}\varphi(y)d\sigma(y), \tag{2.42}$$

and $(\mathcal{K}^{\alpha,\omega})^*$ is given by

$$(\mathcal{K}^{\alpha,\omega})^*\varphi(x) = \text{p.v.} \int_{\partial D} \frac{\partial \mathbf{G}^{\alpha,\omega}(x,y)}{\partial \nu_x}\varphi(y)d\sigma(y). \tag{2.43}$$

Note that $(\mathcal{K}^{\alpha,\omega})^*$ is the L^2-adjoint of $\mathcal{K}^{\alpha,\omega}$. The formulae (2.40) and (2.41) hold because $\mathbf{G}^{\alpha,\omega}(x,y)$ has the same kind of singulary at $x = y$ as that of $\mathbf{\Gamma}(x-y)$.

The following lemma will be of use in later sections

Lemma 2.7 *For any constant vector ϕ*

$$(\frac{1}{2}I + \mathcal{K}^{0,0})\phi = |Y \setminus D|\phi \quad \text{on } \partial D, \tag{2.44}$$

and for any $\psi \in L^2(\partial D)^2$

$$\int_{\partial D} \left(\frac{1}{2}I + (\mathcal{K}^{0,0})^*\right)\psi = |Y \setminus D| \int_{\partial D} \psi. \tag{2.45}$$

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Proof. By Green's theorem and (2.39) we have

$$\mathcal{D}^{0,0}\phi(x) = \int_D \mathcal{L}^{\lambda,\mu} \mathbf{G}^{0,0}(x,y)\phi dy = \phi - \int_D \phi,$$

and hence (2.44) follows since $(\frac{1}{2}I + \mathcal{K}^{0,0})\phi = \mathcal{D}^{0,0}\phi|_-$.

The identity (2.45) immediately follows from (2.44). In fact, for any constant vector ϕ, we have

$$\int_{\partial D} \phi \cdot \left(\frac{1}{2}I + (\mathcal{K}^{0,0})^*\right)\psi = \int_{\partial D} \left(\frac{1}{2}I + \mathcal{K}^{0,0}\right)\phi \cdot \psi$$
$$= |Y \setminus D| \int_{\partial D} \phi \cdot \psi.$$

Thus (2.45) follows. □

3 Asymptotic behavior of phononic bands

The phononic crystal we consider in this paper is a homogeneous elastic medium of Lamé constants λ, μ which contains an array of arbitrary-shaped inclusions $\Omega = \cup_{n\in\mathbb{Z}^2}(D+n)$ which is periodic with respect to the lattice $\mathbb{Z}^2$. These inclusions have Lamé constants $\tilde{\lambda}$, $\tilde{\mu}$. Let $Y =]0,1[^2$ denote the fundamental period cell. For each quasi-momentum $\alpha \in [0, 2\pi[^2$, let $\sigma_\alpha(D)$ be the (discrete) spectrum of the operator defined by (1.1) with the condition that $e^{-\sqrt{-1}\alpha\cdot x}\mathbf{u}$ is periodic. In other words, $\sigma_\alpha(D)$ is the spectrum of the problem

$$\begin{cases} \mathcal{L}^{\lambda,\mu}\mathbf{u} + \omega^2\mathbf{u} = 0, & \text{in } Y \setminus \overline{D}, \\ \mathcal{L}^{\tilde{\lambda},\tilde{\mu}}\mathbf{u} + \omega^2\mathbf{u} = 0, & \text{in } D, \\ \mathbf{u}\big|_+ - \mathbf{u}\big|_- = 0, & \text{on } \partial D, \\ \dfrac{\partial\mathbf{u}}{\partial\nu}\Big|_+ - \dfrac{\partial\mathbf{u}}{\partial\tilde{\nu}}\Big|_- = 0, & \text{on } \partial D, \\ e^{-\sqrt{-1}\alpha\cdot x}\mathbf{u} \text{ is periodic.} \end{cases} \tag{3.1}$$

Here $\mathcal{L}^{\tilde{\lambda},\tilde{\mu}}$ is the elastostatic system corresponding to the Lamé constants $\tilde{\lambda}$ and $\tilde{\mu}$ and $\partial/\partial\tilde{\nu}$ denotes the corresponding conormal derivative.

By the standard Floquet theory, the spectrum of (3.1) has the band structure given by

$$\bigcup_{\alpha\in[0,2\pi[^2} \sigma_\alpha(D). \tag{3.2}$$

The main objective of this section is to investigate the behavior of $\sigma_\alpha(D)$ as $\mu \to +\infty$.

3.1 Integral representation of quasi-periodic solutions

In this section, we obtain the integral representation formula for the solution to (3.1). We denote by $\tilde{\mathcal{S}}^\omega, \tilde{\mathcal{D}}^\omega$, and $\tilde{\mathcal{K}}^\omega$ the layer potentials associated with the Lamé parameters $(\tilde{\lambda}, \tilde{\mu})$.

We first prove the following lemma.

Lemma 3.1 *Suppose that ω^2 is not an eigenvalue for $-\mathcal{L}^{\lambda,\mu}$ in D with the Dirichlet boundary condition on ∂D. Let $\mathbf{u}$ be a solution to (3.1). Then we have*

$$\mathbf{u}|_{\partial D} \perp \operatorname{Ker} \tilde{\mathcal{S}}^\omega \quad \text{and} \quad \mathbf{u}|_{\partial D} \perp \operatorname{Ker}(\mathcal{S}^{\alpha,\omega})^*.$$

Here $\tilde{\mathcal{S}}^\omega$ and $\mathcal{S}^{\alpha,\omega}$ are considered to be operators on $L^2(\partial D)^2$.

Proof. We first observe that, since $(\mathcal{L}^{\tilde\lambda,\tilde\mu} + \omega^2)\mathbf{u} = 0$ in D, we have

$$\mathbf{u}(x) = \tilde{\mathcal{D}}^\omega \left(\mathbf{u}|_{\partial D}\right)(x) - \tilde{\mathcal{S}}^\omega \left(\left.\frac{\partial \mathbf{u}}{\partial \tilde\nu}\right|_-\right)(x), \quad x \in D, \tag{3.3}$$

and consequently by (2.20)

$$\frac{1}{2}\mathbf{u}|_{\partial D} = \tilde{\mathcal{K}}^\omega \left(\mathbf{u}|_{\partial D}\right) - \tilde{\mathcal{S}}^\omega \left(\left.\frac{\partial \mathbf{u}}{\partial \tilde\nu}\right|_-\right). \tag{3.4}$$

Let $\phi \in \operatorname{Ker}(\tilde{\mathcal{S}}^\omega)$, *i.e.*, $\tilde{\mathcal{S}}^\omega \phi = 0$ on ∂D. By Lemma 2.5, we have $\tilde{\mathcal{S}}^\omega \phi = 0$ in $\mathbb{R}^2 \setminus D$ and hence $\frac{1}{2}\phi + (\tilde{\mathcal{K}}^\omega)^* \phi = 0$ by (2.19). Then we have from (3.4)

$$\begin{aligned}
\frac{1}{2}\langle \mathbf{u}|_{\partial D}, \phi \rangle &= \left\langle \tilde{\mathcal{K}}^\omega \left(\mathbf{u}|_{\partial D}\right), \phi \right\rangle - \left\langle \tilde{\mathcal{S}}^\omega \left(\left.\frac{\partial \mathbf{u}}{\partial \tilde\nu}\right|_-\right), \phi \right\rangle \\
&= \left\langle \mathbf{u}|_{\partial D}, (\tilde{\mathcal{K}}^\omega)^* \phi \right\rangle - \left\langle \left.\frac{\partial \mathbf{u}}{\partial \tilde\nu}\right|_-, \tilde{\mathcal{S}}^\omega \phi \right\rangle \\
&= -\frac{1}{2}\langle \mathbf{u}|_{\partial D}, \phi \rangle,
\end{aligned}$$

which implies $\langle \mathbf{u}|_{\partial D}, \phi \rangle = 0$, and hence $\mathbf{u}|_{\partial D} \perp \operatorname{Ker} \tilde{\mathcal{S}}^\omega$.

Observe that if $\mathbf{u}$ is α-quasi-periodic, then

$$\mathcal{D}_Y^{\alpha,\omega} \left(\mathbf{u}|_{\partial Y}\right) = 0 \quad \text{and} \quad \mathcal{S}_Y^{\alpha,\omega} \left(\left.\frac{\partial \mathbf{u}}{\partial \nu}\right|_+\right) = 0 \quad \text{on } \partial Y,$$

where $\mathcal{D}_Y^{\alpha,\omega}$ and $\mathcal{S}_Y^{\alpha,\omega}$ are the (α-quasi-periodic) double and single layer potentials on ∂Y. Thus we have

$$\mathbf{u}(x) = -\mathcal{D}^{\alpha,\omega} \left(\mathbf{u}|_{\partial D}\right)(x) + \mathcal{S}^{\alpha,\omega} \left(\left.\frac{\partial \mathbf{u}}{\partial \nu}\right|_+\right)(x), \quad x \in Y \setminus \overline{D},$$

and consequently,

$$\frac{1}{2}\mathbf{u}|_{\partial D} = -\mathcal{K}^{-\alpha,\omega} \left(\mathbf{u}|_{\partial D}\right) + \mathcal{S}^{\alpha,\omega} \left(\left.\frac{\partial \mathbf{u}}{\partial \nu}\right|_+\right).$$

Let $\phi \in \operatorname{Ker}(\mathcal{S}^{\alpha,\omega})^*$. Since $(\mathcal{S}^{\alpha,\omega})^* = \mathcal{S}^{-\alpha,\omega}$, we have

$$\mathcal{S}^{-\alpha,\omega} \phi = 0 \quad \text{on } \partial D.$$

Since ω^2 is not a Dirichlet eigenvalue of $-\mathcal{L}^{\lambda,\mu}$ in D, we immediately deduce that

$$\mathcal{S}^{-\alpha,\omega} \phi = 0 \quad \text{in } D,$$

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and hence

$$-\frac{1}{2}\phi + (\mathcal{K}^{-\alpha,\omega})^*\phi = 0 \quad \text{on } \partial D.$$

Therefore, we get

$$\frac{1}{2}\langle \mathbf{u}|_{\partial D}, \phi \rangle = -\langle \mathcal{K}^{-\alpha,\omega}(\mathbf{u}|_{\partial D}), \phi \rangle + \left\langle \mathcal{S}^{\alpha,\omega}\left(\frac{\partial \mathbf{u}}{\partial \nu}\Big|_+\right), \phi \right\rangle$$

$$= -\left\langle \mathbf{u}|_{\partial D}, (\mathcal{K}^{-\alpha,\omega})^*\phi \right\rangle + \left\langle \frac{\partial \mathbf{u}}{\partial \nu}\Big|_+, \mathcal{S}^{-\alpha,\omega}\phi \right\rangle$$

$$= -\frac{1}{2}\langle \mathbf{u}|_{\partial D}, \phi \rangle,$$

which implies $\langle \mathbf{u}|_{\partial D}, \phi \rangle = 0$. This completes the proof. $\qquad \square$

We now establish a representation formula for solutions of (3.1).

Theorem 3.2 *Suppose that ω^2 is not an eigenvalue for $-\mathcal{L}^{\lambda,\mu}$ in D with the Dirichlet boundary condition on ∂D. Then, for any solution $\mathbf{u}$ of (3.1), there exists one and only one pair $(\phi, \psi) \in L^2(\partial D)^2 \times L^2(\partial D)^2$ such that*

$$\mathbf{u}(x) = \begin{cases} \tilde{\mathcal{S}}^\omega \phi(x), & x \in D, \\ \mathcal{S}^{\alpha,\omega}\psi(x), & x \in Y \setminus \overline{D}. \end{cases} \tag{3.5}$$

Moreover, (ϕ, ψ) satisfies

$$\begin{cases} \tilde{\mathcal{S}}^\omega \phi - \mathcal{S}^{\alpha,\omega}\psi = 0 & \text{on } \partial D, \\ \left(\frac{1}{2}I - (\tilde{\mathcal{K}}^\omega)^*\right)\phi + \left(\frac{1}{2}I + (\mathcal{K}^{\alpha,\omega})^*\right)\psi = 0 & \text{on } \partial D, \end{cases} \tag{3.6}$$

and the mapping $\mathbf{u} \longmapsto (\phi, \psi)$ from solutions of (3.1) in $H^1(Y)^2$ to solutions to the system of integral equations (3.6) in $L^2(\partial D)^2 \times L^2(\partial D)^2$ is a one-to-one correspondence.

Proof. We first note that the problem of finding (ϕ, ψ) satisfying (3.5) and (3.6) is equivalent to solving the following two systems of equations:

$$\begin{cases} \tilde{\mathcal{S}}^\omega \phi = \mathbf{u}|_{\partial D} & \text{on } \partial D, \\ \left(-\frac{1}{2}I + (\tilde{\mathcal{K}}^\omega)^*\right)\phi = \frac{\partial \mathbf{u}}{\partial \tilde{\nu}}\Big|_- & \text{on } \partial D, \end{cases} \tag{3.7}$$

and

$$\begin{cases} \mathcal{S}^{\alpha,\omega}\psi = \mathbf{u}|_{\partial D} & \text{on } \partial D, \\ \left(\frac{1}{2}I + (\mathcal{K}^{\alpha,\omega})^*\right)\psi = \frac{\partial \mathbf{u}}{\partial \nu}\Big|_+ & \text{on } \partial D. \end{cases} \tag{3.8}$$

In order to find ϕ satisfying (3.7), it suffices to find ϕ satisfying $\tilde{\mathcal{S}}^\omega \phi = \mathbf{u}$ in D. Since $\tilde{\mathcal{S}}^\omega$ is self-adjoint Fredholm operator on $L^2(\partial D)^2$ with index 0 [2], it follows from Lemma 3.1 that there exists $\phi_0 \in L^2(\partial D)^2$ such that

$$\tilde{\mathcal{S}}^\omega \phi_0 = \mathbf{u}|_{\partial D} \quad \text{on } \partial D. \tag{3.9}$$

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Observe that if $\omega \neq 0$, then the solution to the Dirichlet problem for $\mathcal{L}^{\lambda,\mu} + \omega^2$ may not be unique, and hence (3.9) does not imply $\tilde{\mathcal{S}}^\omega \phi_0 = \mathbf{u}$ in D. However, since $(\mathcal{L}^{\lambda,\mu} + \omega^2)(\mathbf{u} - \tilde{\mathcal{S}}^\omega \phi_0) = 0$ in D, we get by Green's formula

$$\mathbf{u} - \tilde{\mathcal{S}}^\omega \phi_0 = -\tilde{\mathcal{S}}^\omega \left[\frac{\partial}{\partial \tilde{\nu}} \left(\mathbf{u} - \tilde{\mathcal{S}}^\omega \phi_0 \right) \Big|_{-} \right] \quad \text{in } D,$$

and therefore,

$$\mathbf{u} = \tilde{\mathcal{S}}^\omega \left[\phi_0 - \frac{\partial}{\partial \tilde{\nu}} \left(\mathbf{u} - \tilde{\mathcal{S}}^\omega \phi_0 \right) \Big|_{-} \right] \quad \text{in } D.$$

To prove the uniqueness of ϕ satisfying (3.7), it suffices to show that the solution to

$$\begin{cases} \tilde{\mathcal{S}}^\omega \phi = 0 & \text{on } \partial D, \\ \left(-\frac{1}{2} I + (\tilde{\mathcal{K}}^\omega)^* \right) \phi = 0 & \text{on } \partial D, \end{cases}$$

is zero. By the first equation in (3.7) and Lemma 2.5, $\tilde{\mathcal{S}}^\omega \phi = 0$ in $\mathbb{R}^2 \setminus \overline{D}$ and hence $\phi = \frac{\partial}{\partial \tilde{\nu}} \tilde{\mathcal{S}}^\omega \phi \Big|_{+} - \frac{\partial}{\partial \tilde{\nu}} \tilde{\mathcal{S}}^\omega \phi \Big|_{-} = 0$.

Similarly, we can show existence and uniqueness of ψ satisfying

$$\mathbf{u} = \mathcal{S}^{\alpha,\omega} \psi \quad \text{in } Y \setminus \overline{D},$$

which yields (3.8). This completes the proof. $\square$

Let $\mathcal{A}^{\alpha,\omega}$ be the operator-valued function of ω defined by

$$\mathcal{A}^{\alpha,\omega} := \begin{pmatrix} \tilde{\mathcal{S}}^\omega & -\mathcal{S}^{\alpha,\omega} \\ \frac{1}{2} I - (\tilde{\mathcal{K}}^\omega)^* & \frac{1}{2} I + (\mathcal{K}^{\alpha,\omega})^* \end{pmatrix}. \tag{3.10}$$

By Theorem 3.2, ω^2 is an eigenvalue corresponding to quasi-momentum α if and only if ω is a characteristic value of $\mathcal{A}^{\alpha,\omega}$. Consequently, we have now a new way of computing the spectrum of (3.1) by examining the characteristic values of $\mathcal{A}^{\alpha,\omega}$. Based on Muller's method [31] for finding complex roots of scalar equations, a boundary element method similar to the one developed in [11, 12] can be designed for computing phononic band gaps.

3.2 Full asymptotic expansions

Expanding the operator-valued functions $\mathcal{A}^{\alpha,\omega}$ in terms of μ as $\mu \to +\infty$, we can calculate asymptotic expressions of their characteristic values with the help of the generalized Rouché's theorem, and it is what we do in this subsection.

We begin with the following asymptotic expansion of $G_{ij}^{\alpha,\omega}(x,y)$ in (2.34).

Lemma 3.3 Let $\tau_l = 1 - \left(\frac{c_T}{c_L} \right)^{2l}$. As $\mu \to +\infty$,

$$G_{ij}^{\alpha,\omega}(x,y) = \sum_{l=1}^{+\infty} \frac{\omega^{2(l-1)}}{\mu^l} \sum_{n \in \mathbb{Z}^2} e^{\sqrt{-1}(2\pi n + \alpha) \cdot (x-y)} \left(\frac{-\delta_{ij}}{|2\pi n + \alpha|^{2l}} + \tau_l \frac{(2\pi n + \alpha)_i (2\pi n + \alpha)_j}{|2\pi n + \alpha|^{2(l+1)}} \right),$$
$$\tag{3.11}$$

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for fixed $\alpha \neq 0$, while for $\alpha = 0$,

$$G_{ij}^{0,\omega}(x,y) = \frac{\delta_{ij}}{\omega^2} + \sum_{l=1}^{+\infty} \frac{\omega^{2(l-1)}}{(2\pi)^{2l}\mu^l} \sum_{n \in \mathbb{Z}^2 \backslash \{0\}} e^{2\pi\sqrt{-1}n\cdot(x-y)} \left(-\frac{\delta_{ij}}{|n|^{2l}} + \tau_l \frac{n_i n_j}{|n|^{2(l+1)}} \right). \qquad (3.12)$$

Derivation of (3.11) and (3.12) are straightforward. In fact, since

$$\frac{1}{k_T^2 - |2\pi n + \alpha|^2} = \frac{1}{\frac{\omega^2}{\mu} - |2\pi n + \alpha|^2} = -\sum_{k=0}^{\infty} \frac{\omega^{2k}}{\mu^k |2\pi n + \alpha|^{2(k+1)}},$$

one immediately obtains (3.11) and (3.12).

We can write (3.11) and (3.12) as

$$\mathbf{G}^{\alpha,\omega}(x,y) = \sum_{l=1}^{+\infty} \frac{1}{\mu^l} \mathbf{G}_l^{\alpha,\omega}(x,y), \qquad (3.13)$$

and

$$\mathbf{G}^{0,\omega}(x,y) = \frac{1}{\omega^2}I + \sum_{l=1}^{+\infty} \frac{1}{\mu^l} \mathbf{G}_l^{0,\omega}(x,y), \qquad (3.14)$$

where the definitions of $\mathbf{G}_l^{\alpha,\omega}(x,y)$ and $\mathbf{G}_l^{0,\omega}(x,y)$ are obvious from (3.11) and (3.12). We note that $\mathbf{G}_l^{\alpha,\omega}(x,y)$ and $\mathbf{G}_l^{0,\omega}(x,y)$ are dependent upon μ because of the factor τ_l. However, since $|\tau_l| \leq C$ for some constant C independent of μ and l, it will not affect analysis to follow. We also note that $\mathbf{G}_1^{\alpha,\omega}(x,y)$ is independent of ω and

$$\mathbf{G}_1^{\alpha,\omega}(x,y) = \mu \mathbf{G}^{\alpha,0}(x,y), \qquad (3.15)$$

where $\mathbf{G}^{\alpha,0}(x,y)$ is the quasi-periodic Green function defined in (2.36).

Denote by $\mathcal{S}_l^{\alpha,\omega}$ and $(\mathcal{K}_l^{\alpha,\omega})^*$, for $l \geq 1$ and $\alpha \in [0, 2\pi[^2$, the single layer potential and the boundary integral operator associated with the kernel $\mathbf{G}_l^{\alpha,\omega}(x,y)$ as defined in (2.43) so that

$$\mathcal{S}^{\alpha,\omega} = \sum_{l=1}^{+\infty} \frac{1}{\mu^l} \mathcal{S}_l^{\alpha,\omega} \quad \text{and} \quad (\mathcal{K}^{\alpha,\omega})^* = \sum_{l=1}^{+\infty} \frac{1}{\mu^l} (\mathcal{K}_l^{\alpha,\omega})^*. \qquad (3.16)$$

Lemma 3.4 *The operator $\frac{1}{2}I + (\mathcal{K}^{\alpha,0})^* : L^2(\partial D)^2 \to L^2(\partial D)^2$ is invertible.*

Before proving Lemma 3.4, let us make a note of the following simple fact: If $\mathbf{u}$ and $\mathbf{v}$ are α-quasi-periodic, then

$$\int_{\partial Y} \frac{\partial \mathbf{u}}{\partial \nu} \cdot \overline{\mathbf{v}} d\sigma = 0. \qquad (3.17)$$

To prove this, it is enough to see that

$$\int_{\partial Y} \frac{\partial \mathbf{u}}{\partial \nu} \cdot \overline{\mathbf{v}} = \int_{\partial Y} \frac{\partial(e^{-i\alpha\cdot x}\mathbf{u})}{\partial \nu} \cdot \overline{e^{-i\alpha\cdot x}\mathbf{v}}$$

$$+ i \int_{\partial Y} \left[\lambda\alpha \cdot (e^{-i\alpha\cdot x}\mathbf{u})N + \mu \begin{pmatrix} 2\alpha_1 N_1 + \alpha_2 N_2 & \alpha_1 N_2 \\ \alpha_2 N_1 & \alpha_1 N_1 + 2\alpha_2 N_2 \end{pmatrix} (e^{-i\alpha\cdot x}\mathbf{u}) \right] \cdot \overline{e^{-i\alpha\cdot x}\mathbf{v}}.$$

Here $i = \sqrt{-1}$ and N is the outward unit normal to Y. Then the integrands over the opposite sides of ∂Y have the same absolute values with different signs and therefore the integration over ∂Y is zero.

Proof of Lemma 3.4. For $\alpha \neq 0$, we show injectivity of $\frac{1}{2}I + (\mathcal{K}^{\alpha,0})^*$. Then from the Fredholm alternative, the result follows. Suppose $\phi \in L^2(\partial D)^2$ satisfies

$$(\frac{1}{2}I + (\mathcal{K}^{\alpha,0})^*)\phi = 0 \quad \text{on } \partial D.$$

Then by (2.40), $\mathbf{u} := \mathcal{S}^{\alpha,0}\phi$ satisfies

$$\begin{cases} \mathcal{L}^{\lambda,\mu}\mathbf{u} = 0 & \text{in } Y \setminus \overline{D}, \\ \left.\dfrac{\partial \mathbf{u}}{\partial \nu}\right|_+ = 0 & \text{on } \partial D, \\ \mathbf{u} \text{ is } \alpha\text{-quasi-periodic.} \end{cases}$$

Therefore, it follows from (3.17) that

$$\int_{Y \setminus D} \left(\lambda |\nabla \cdot \mathbf{u}|^2 + \frac{\mu}{2}|\nabla \mathbf{u} + \nabla \mathbf{u}^t|^2 \right) = \int_{\partial Y} \frac{\partial \mathbf{u}}{\partial \nu} \cdot \overline{\mathbf{u}} - \int_{\partial D} \left.\frac{\partial \mathbf{u}}{\partial \nu}\right|_+ \cdot \overline{\mathbf{u}} = 0.$$

Thus, $\mathbf{u}$ is constant in $Y \setminus \overline{D}$, and hence in D. Hence, we get

$$\phi = \left.\frac{\partial \mathbf{u}}{\partial \nu}\right|_+ - \left.\frac{\partial \mathbf{u}}{\partial \nu}\right|_- = 0.$$

For the periodic case $\alpha = 0$, we show the injectivity of $\frac{1}{2}I + \mathcal{K}^{0,0}$. Let $\phi \in L^2(\partial D)^2$ satisfying $(\frac{1}{2}I + \mathcal{K}^{0,0})\phi = 0$ on ∂D. Then $\mathbf{u} := \mathcal{D}^{0,0}\phi$ satisfies

$$\begin{cases} \mathcal{L}^{\lambda,\mu}\mathbf{u} = 0 & \text{in } D, \\ \mathbf{u}|_- = 0 & \text{on } \partial D, \end{cases}$$

and therefore $\mathbf{u} = 0$ in D. Furthermore, if $(\frac{1}{2}I + \mathcal{K}^{0,0})\phi = 0$, we can show that $\phi \in H^1(\partial D)^2$ and $\left.\dfrac{\partial(\mathcal{D}^{0,0}\phi)}{\partial \nu}\right|_+ = \left.\dfrac{\partial(\mathcal{D}^{0,0}\phi)}{\partial \nu}\right|_-$. See [2] for the details. Then we have

$$\begin{cases} \mathcal{L}^{\lambda,\mu}\mathbf{u} = 0 & \text{in } Y \setminus \overline{D}, \\ \left.\dfrac{\partial \mathbf{u}}{\partial \nu}\right|_+ = 0 & \text{on } \partial D, \\ \mathbf{u} \text{ is periodic.} \end{cases}$$

Therefore, it follows that

$$\int_{Y \setminus D} \left(\lambda |\nabla \cdot \mathbf{u}|^2 + \frac{\mu}{2}|\nabla \mathbf{u} + \nabla \mathbf{u}^t|^2 \right) = \int_{\partial Y} \frac{\partial \mathbf{u}}{\partial \nu} \cdot \overline{\mathbf{u}} - \int_{\partial D} \left.\frac{\partial \mathbf{u}}{\partial \nu}\right|_+ \cdot \overline{\mathbf{u}} = 0.$$

Thus, $\mathbf{u}$ is constant in $Y \setminus \overline{D}$, and hence $\phi = \mathbf{u}|_- - \mathbf{u}|_+$ is constant. By (2.44), we obtain that

$$0 = (\frac{1}{2}I + \mathcal{K}^{0,0})\phi = |Y \setminus D|\phi,$$

which implies that ϕ must be zero. This completes the proof. $\qquad\square$

We now derive complete asymptotic expansions of eigenvalues as $\mu \to +\infty$. We deal with three cases separately: $\alpha \neq 0$ (not of order $O(\frac{1}{\sqrt{\mu}})$), $\alpha = 0$, and $|\alpha|$ is of order $O(\frac{1}{\sqrt{\mu}})$

3.2.1 The case $\alpha \neq 0$.

The following lemma, which is an immediate consequence of (3.16), gives a complete asymptotic expansion of $\mathcal{A}^{\alpha,\omega}$ defined in (3.10) as $\mu \to +\infty$.

Lemma 3.5 *Suppose $\alpha \neq 0$. Let*

$$\mathcal{A}_0^{\alpha,\omega} = \begin{pmatrix} \tilde{\mathcal{S}}^\omega & 0 \\ \frac{1}{2}I - (\tilde{\mathcal{K}}^\omega)^* & \frac{1}{2}I + (\mathcal{K}^{\alpha,0})^* \end{pmatrix}, \tag{3.18}$$

and, for $l \geq 1$,

$$\mathcal{A}_l^{\alpha,\omega} = \begin{pmatrix} 0 & -\mathcal{S}_l^{\alpha,\omega} \\ 0 & \frac{1}{\mu}(\mathcal{K}_{l+1}^{\alpha,\omega})^* \end{pmatrix}. \tag{3.19}$$

Then we have

$$\mathcal{A}^{\alpha,\omega} = \mathcal{A}_0^{\alpha,\omega} + \sum_{l=1}^{+\infty} \frac{1}{\mu^l} \mathcal{A}_l^{\alpha,\omega}. \tag{3.20}$$

All the operators are defined on $L^2(\partial D)^2 \times L^2(\partial D)^2$.

Note that it is just for convenience that there is $1/\mu$ in the definition of $\mathcal{A}_l^{\alpha,\omega}$. This of course does not affect any of our asymptotic results.

Lemma 3.6 *Suppose $\alpha \neq 0$. Then the followings are equivalent:*

(i) *$\omega_0^\alpha \in \mathbb{R}$ is a characteristic value of $\mathcal{A}_0^{\alpha,\omega}$,*

(ii) *$\omega_0^\alpha \in \mathbb{R}$ is a characteristic value of $\tilde{\mathcal{S}}^\omega$,*

(iii) *$(\omega_0^\alpha)^2$ is an eigenvalue of $-\mathcal{L}^{\tilde{\lambda},\tilde{\mu}}$ in D with the Dirichlet boundary condition.*

Moreover if $\mathbf{u}$ is an eigenfunction of $-\mathcal{L}^{\tilde{\lambda},\tilde{\mu}}$ in D with the Dirichlet boundary condition, then $\varphi := \frac{\partial \mathbf{u}}{\partial \nu}|_-$ is an characteristic function of $\tilde{\mathcal{S}}^\omega$. Conversely, if φ is an characteristic function of $\tilde{\mathcal{S}}^\omega$, then $\mathbf{u} := -\tilde{\mathcal{S}}^\omega(\varphi)$ is an eigenfunction of $-\mathcal{L}^{\tilde{\lambda},\tilde{\mu}}$ in D with the Dirichlet boundary condition.

Proof. By Lemma 3.4, $\frac{1}{2}I + (\mathcal{K}^{\alpha,0})^*$ is invertible. Thus characteristic values of $\mathcal{A}_0^{\alpha,\omega}$ coincide with those of $\tilde{\mathcal{S}}^\omega$. On the other hand, the Green identity (3.3) shows that the characteristic values of $\tilde{\mathcal{S}}^\omega$ are exactly eigenvalues of $-\mathcal{L}^{\tilde{\lambda},\tilde{\mu}}$ in D with the Dirichlet boundary condition. The last statements of Lemma 3.6 also follow from (3.3). $\square$

Lemma 3.7 *Every eigenvector of $\tilde{\mathcal{S}}^\omega$ has rank one.*

Proof. Let ϕ be an eigenvector of $\tilde{\mathcal{S}}^\omega$ associated with the characteristic value ω_0, *i.e.*, $\tilde{\mathcal{S}}^{\omega_0}\phi = 0$ on ∂D. Suppose that there exists ϕ^ω, holomorphic in a neighborhood of ω_0 as a function of ω, such that $\phi^{\omega_0} = \phi$ and

$$\tilde{\mathcal{S}}^\omega \phi^\omega = (\omega^2 - \omega_0^2)\psi^\omega$$

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for some ψ^ω. Let $\mathbf{u}^\omega(x) := \tilde{\mathcal{S}}^\omega \phi^\omega(x)$, $x \in D$. Then $\mathbf{u}^\omega$ satisfies

$$
\begin{cases}
(\mathcal{L}^{\tilde{\lambda},\tilde{\mu}} + \omega^2)\mathbf{u}^\omega = 0 & \text{in } D, \\
\mathbf{u}^\omega = (\omega^2 - \omega_0^2)\psi^\omega & \text{on } \partial D.
\end{cases}
$$

By Green's formula, we have

$$
(\omega^2 - \omega_0^2) \int_D \mathbf{u}^\omega \cdot \overline{\mathbf{u}^{\omega_0}} = \int_D \mathbf{u}^\omega \cdot \overline{\mathcal{L}^{\tilde{\lambda},\tilde{\mu}}\mathbf{u}^{\omega_0}} - \mathcal{L}^{\tilde{\lambda},\tilde{\mu}}\mathbf{u}^\omega \cdot \overline{\mathbf{u}^{\omega_0}}
$$

$$
= \int_{\partial D} \mathbf{u}^\omega \cdot \frac{\partial \overline{\mathbf{u}^{\omega_0}}}{\partial \tilde{\nu}} = (\omega^2 - \omega_0^2) \int_{\partial D} \psi^\omega \cdot \frac{\partial \overline{\mathbf{u}^{\omega_0}}}{\partial \tilde{\nu}}.
$$

Dividing by $\omega^2 - \omega_0^2$ and letting $\omega \to \omega_0$, we have

$$
\int_D |\mathbf{u}^{\omega_0}|^2 = \int_{\partial D} \psi^{\omega_0} \cdot \frac{\partial \overline{\mathbf{u}^{\omega_0}}}{\partial \tilde{\nu}}.
$$

Therefore we conclude that ψ^{ω_0} is not identically zero. This completes the proof. $\qquad\square$

By Lemma 3.4 and the fact that $\tilde{\mathcal{S}}^\omega$ is Fredholm, we know that $\mathcal{A}_0^{\alpha,\omega}$ is normal. Moreover, Lemma 3.7 says that the multiplicity of $\mathcal{A}_0^{\alpha,\omega}$ at each eigenvalue ω_0^2 of $-\mathcal{L}^{\tilde{\lambda},\tilde{\mu}}$ is equal to the dimension of $\operatorname{Ker} \tilde{\mathcal{S}}^{\omega_0}$. Combining this fact with Theorem 2.2, we obtain the following lemma.

Lemma 3.8 *For each eigenvalue ω_0^2 of $-\mathcal{L}^{\tilde{\lambda},\tilde{\mu}}$ and sufficiently large μ, there exists a small neighborhood V of $\omega_0 > 0$ such that $\mathcal{A}^{\alpha,\omega}$ is normal with respect to ∂V and $\mathcal{M}(\mathcal{A}^{\alpha,\omega}, \partial V) = \dim \operatorname{Ker} \tilde{\mathcal{S}}^{\omega_0}$.*

Let ω_0^2 (with $\omega_0 > 0$) be a simple eigenvalue of $-\mathcal{L}^{\tilde{\lambda},\tilde{\mu}}$ in D with the Dirichlet boundary condition. There exists a unique eigenvalue $(\omega_\mu^\alpha)^2$ (with $\omega_\mu^\alpha > 0$) of (3.1) lying in a small complex neighborhood V of ω_0. Combining the generalized Rouché's theorem with Lemma 3.5 we are now able to derive complete asymptotic formulae for the characteristic values of $\omega \mapsto \mathcal{A}^{\alpha,\omega}$. Theorem 2.4 yields that

$$
\omega_\mu^\alpha - \omega_0 = \frac{1}{2\sqrt{-1}\pi} \operatorname{tr} \int_{\partial V} (\omega - \omega_0)(\mathcal{A}^{\alpha,\omega})^{-1} \frac{d}{d\omega}\mathcal{A}^{\alpha,\omega} d\omega. \tag{3.21}
$$

Then we obtain the following complete asymptotic expansion for the eigenvalue perturbations $\omega_\mu^\alpha - \omega_0$.

Theorem 3.9 *Suppose $\alpha \neq 0$. Then the following asymptotic expansion holds:*

$$
\omega_\mu^\alpha - \omega_0 = \frac{1}{2\pi\sqrt{-1}} \sum_{p=1}^{+\infty} \frac{1}{p} \sum_{n=p}^{+\infty} \frac{1}{\mu^n} \operatorname{tr} \int_{\partial V} \mathcal{B}_{n,p}^{\alpha,\omega} d\omega, \tag{3.22}
$$

where

$$
\mathcal{B}_{n,p}^{\alpha,\omega} = (-1)^p \sum_{\substack{n_1 + \cdots + n_p = n \\ n_i \geq 1}} (\mathcal{A}_0^{\alpha,\omega})^{-1} \mathcal{A}_{n_1}^{\alpha,\omega} \cdots (\mathcal{A}_0^{\alpha,\omega})^{-1} \mathcal{A}_{n_p}^{\alpha,\omega}. \tag{3.23}
$$

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Proof. For sufficiently large μ, the following Neumann series converges uniformly with respect to the variable $\omega \in \partial V$:

$$(\mathcal{A}^{\alpha,\omega})^{-1} = \sum_{p=0}^{+\infty} \left[(\mathcal{A}_0^{\alpha,\omega})^{-1}(\mathcal{A}_0^{\alpha,\omega} - \mathcal{A}^{\alpha,\omega}) \right]^p (\mathcal{A}_0^{\alpha,\omega})^{-1}.$$

By (2.10) and the relation

$$\frac{d}{d\omega}(\mathcal{A}_0^{\alpha,\omega})^{-1} = -(\mathcal{A}_0^{\alpha,\omega})^{-1}\frac{d}{d\omega}\mathcal{A}_0^{\alpha,\omega}(\mathcal{A}_0^{\alpha,\omega})^{-1},$$

we get

$$\frac{1}{2\pi\sqrt{-1}}\, \text{tr} \int_{\partial V} (\omega - \omega_0)\frac{1}{p}\frac{d}{d\omega}\left[(\mathcal{A}_0^{\alpha,\omega})^{-1}(\mathcal{A}_0^{\alpha,\omega} - \mathcal{A}^{\alpha,\omega}) \right]^p d\omega$$

$$= \frac{1}{2\pi\sqrt{-1}}\, \text{tr} \int_{\partial V} (\omega - \omega_0)\left[(\mathcal{A}_0^{\alpha,\omega})^{-1}(\mathcal{A}_0^{\alpha,\omega} - \mathcal{A}^{\alpha,\omega}) \right]^{p-1} (\mathcal{A}_0^{\alpha,\omega})^{-1}\frac{d}{d\omega}(\mathcal{A}_0^{\alpha,\omega} - \mathcal{A}^{\alpha,\omega})d\omega$$

$$- \frac{1}{2\pi\sqrt{-1}}\, \text{tr} \int_{\partial V} (\omega - \omega_0)\left[(\mathcal{A}_0^{\alpha,\omega})^{-1}(\mathcal{A}_0^{\alpha,\omega} - \mathcal{A}^{\alpha,\omega}) \right]^p (\mathcal{A}_0^{\alpha,\omega})^{-1}\frac{d}{d\omega}\mathcal{A}_0^{\alpha,\omega}d\omega.$$

Summing over p, we obtain

$$\frac{1}{2\pi\sqrt{-1}}\sum_{p=1}^{+\infty} \text{tr} \int_{\partial V} (\omega - \omega_0)\frac{1}{p}\frac{d}{d\omega}\left[(\mathcal{A}_0^{\alpha,\omega})^{-1}(\mathcal{A}_0^{\alpha,\omega} - \mathcal{A}^{\alpha,\omega}) \right]^p d\omega$$

$$= -\frac{1}{2\pi\sqrt{-1}}\sum_{p=0}^{+\infty} \text{tr} \int_{\partial V} (\omega - \omega_0)\left[(\mathcal{A}_0^{\alpha,\omega})^{-1}(\mathcal{A}_0^{\alpha,\omega} - \mathcal{A}^{\alpha,\omega}) \right]^p (\mathcal{A}_0^{\alpha,\omega})^{-1}\frac{d}{d\omega}\mathcal{A}^{\alpha,\omega}d\omega$$

$$+ \frac{1}{2\pi\sqrt{-1}}\, \text{tr} \int_{\partial V} (\omega - \omega_0)(\mathcal{A}_0^{\alpha,\omega})^{-1}\frac{d}{d\omega}\mathcal{A}_0^{\alpha,\omega}d\omega.$$

Since

$$\frac{1}{2\pi\sqrt{-1}}\, \text{tr} \int_{\partial V} (\omega - \omega_0)(\mathcal{A}_0^{\alpha,\omega})^{-1}\frac{d}{d\omega}\mathcal{A}_0^{\alpha,\omega}d\omega = 0,$$

and

$$\frac{1}{2\pi\sqrt{-1}}\, \text{tr} \int_{\partial V} (\omega - \omega_0)\frac{d}{d\omega}\left[(\mathcal{A}_0^{\alpha,\omega})^{-1}(\mathcal{A}_0^{\alpha,\omega} - \mathcal{A}^{\alpha,\omega}) \right]^p d\omega$$

$$= -\frac{1}{2\pi\sqrt{-1}}\, \text{tr} \int_{\partial V} \left[(\mathcal{A}_0^{\alpha,\omega})^{-1}(\mathcal{A}_0^{\alpha,\omega} - \mathcal{A}^{\alpha,\omega}) \right]^p d\omega,$$

we have

$$\frac{1}{2\pi\sqrt{-1}}\sum_{p=1}^{+\infty} \text{tr} \int_{\partial V} \frac{1}{p}\left[(\mathcal{A}_0^{\alpha,\omega})^{-1}(\mathcal{A}_0^{\alpha,\omega} - \mathcal{A}^{\alpha,\omega}) \right]^p d\omega$$

$$= \frac{1}{2\pi\sqrt{-1}}\sum_{p=0}^{+\infty} \text{tr} \int_{\partial V} (\omega - \omega_0)\left[(\mathcal{A}_0^{\alpha,\omega})^{-1}(\mathcal{A}_0^{\alpha,\omega} - \mathcal{A}^{\alpha,\omega}) \right]^p (\mathcal{A}_0^{\alpha,\omega})^{-1}\frac{d}{d\omega}\mathcal{A}^{\alpha,\omega}d\omega.$$

$$= \frac{1}{2\pi\sqrt{-1}}\, \text{tr} \int_{\partial V} (\omega - \omega_0)(\mathcal{A}^{\alpha,\omega})^{-1}\frac{d}{d\omega}\mathcal{A}^{\alpha,\omega}d\omega.$$

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By expanding $\left[(\mathcal{A}_0^{\alpha,\omega})^{-1}(\mathcal{A}_0^{\alpha,\omega} - \mathcal{A}^{\alpha,\omega})\right]^p$, we obtain the desired result. $\qquad\square$

3.2.2 The case $\alpha = 0$.

We now deal with the periodic case ($\alpha = 0$). By (3.12) we have

$$\mathcal{A}^{0,\omega} = \mathcal{A}_0^{0,\omega} + \sum_{l=1}^{+\infty} \frac{1}{\mu^l} \mathcal{A}_l^{0,\omega}, \tag{3.24}$$

where

$$\mathcal{A}_0^{0,\omega} = \begin{pmatrix} \tilde{\mathcal{S}}^\omega & -\dfrac{1}{\omega^2} \displaystyle\int_{\partial D} \cdot\, d\sigma \\[2mm] \dfrac{1}{2}I - (\tilde{\mathcal{K}}^\omega)^* & \dfrac{1}{2}I + (\mathcal{K}^{0,0})^* \end{pmatrix}, \tag{3.25}$$

and, for $l \geq 1$,

$$\mathcal{A}_l^{0,\omega} = \begin{pmatrix} 0 & -\mathcal{S}_l^{0,\omega} \\[2mm] 0 & \frac{1}{\mu}(\mathcal{K}_{l+1}^{0,\omega})^* \end{pmatrix}. \tag{3.26}$$

Here we consider the following eigenvalue problem

$$\begin{cases} (\mathcal{L}^{\tilde{\lambda},\tilde{\mu}} + \omega^2)\mathbf{u} = 0 & \text{in } D, \\[2mm] \mathbf{u} + \dfrac{1}{|Y \setminus D|} \displaystyle\int_D \mathbf{u} = 0 & \text{on } \partial D. \end{cases} \tag{3.27}$$

We note that it has a discrete spectrum and its eigenvalues are nonnegative since we have

$$\int_D \tilde{\lambda}|\nabla \cdot \mathbf{u}|^2 + \frac{\tilde{\mu}}{2}|\nabla\mathbf{u} + \nabla\mathbf{u}^t|^2 = \int_{\partial D} \mathbf{u} \cdot \frac{\partial \overline{\mathbf{u}}}{\partial \tilde{\nu}} - \int_D \mathbf{u} \cdot \mathcal{L}^{\tilde{\lambda},\tilde{\mu}}\overline{\mathbf{u}}$$

$$= -\frac{1}{|Y \setminus D|} \int_D \mathbf{u} \cdot \int_{\partial D} \frac{\partial \overline{\mathbf{u}}}{\partial \tilde{\nu}} + \overline{\omega}^2 \int_D |\mathbf{u}|^2$$

$$= \frac{\overline{\omega}^2}{|Y \setminus D|} \left|\int_D \mathbf{u}\right|^2 + \overline{\omega}^2 \int_D |\mathbf{u}|^2.$$

The eigenvalue of (3.27) is related with the characteristic value of $\mathcal{A}^{0,\omega}$ as follows.

Lemma 3.10 *The equation (3.27) has a nonzero solution if and only if ω is a characteristic value of the operator-valued function $\mathcal{A}_0^{0,\omega}$.*

Proof. Suppose that there exists a nonzero pair (ϕ, ψ) such that

$$\mathcal{A}_0^{0,\omega} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0,$$

or equivalently

$$\tilde{\mathcal{S}}^\omega \phi - \frac{1}{\omega^2} \int_{\partial D} \psi\, d\sigma = 0 \quad \text{on } \partial D, \tag{3.28}$$

$$\left(\frac{1}{2}I - (\tilde{\mathcal{K}}^\omega)^*\right)\phi + \left(\frac{1}{2}I + (\mathcal{K}^{0,0})^*\right)\psi = 0 \quad \text{on } \partial D. \tag{3.29}$$

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In particular, ϕ is nonzero by the invertibility of $\frac{1}{2}I + (\mathcal{K}^{0,0})^*$. Let $\mathbf{u} := \tilde{\mathcal{S}}^\omega \phi$. Then we have

$$
\begin{aligned}
\frac{1}{|Y \setminus D|} \int_D \mathbf{u} &= -\frac{1}{\omega^2 |Y \setminus D|} \int_{\partial D} \frac{\partial \mathbf{u}}{\partial \tilde{\nu}} \\
&= -\frac{1}{\omega^2 |Y \setminus D|} \int_{\partial D} \left(-\frac{1}{2}I + (\tilde{\mathcal{K}}^\omega)^* \right) \phi \\
&= -\frac{1}{\omega^2 |Y \setminus D|} \int_{\partial D} \left(\frac{1}{2}I + (\mathcal{K}^{0,0})^* \right) \psi \\
&= -\frac{1}{\omega^2} \int_{\partial D} \psi,
\end{aligned}
$$

where the last equality follows from (2.45). Therefore by (3.28) $\mathbf{u}$ is a nonzero solution to (3.27).

Suppose that (3.27) has nonzero solution $\mathbf{u}$. Following the same argument as in the proof of Theorem 3.2, we can see that there exists ϕ such that

$$
\begin{cases}
\tilde{\mathcal{S}}^\omega \phi = \mathbf{u}|_{\partial D} & \text{on } \partial D, \\
\left(-\frac{1}{2}I + (\tilde{\mathcal{K}}^\omega)^* \right)\phi = \dfrac{\partial \mathbf{u}}{\partial \tilde{\nu}} & \text{on } \partial D.
\end{cases}
\tag{3.30}
$$

If we set

$$
\psi = \left(\frac{1}{2}I + (\mathcal{K}^{0,0})^* \right)^{-1} \left(\frac{\partial \mathbf{u}}{\partial \tilde{\nu}} \right),
$$

then (ϕ, ψ) satisfies

$$
\mathcal{A}_0^{0,\omega} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0.
$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We also have the following Lemma.

Lemma 3.11 *Every eigenvector of $\mathcal{A}_0^{0,\omega}$ has rank one.*

Proof. Suppose that $\begin{pmatrix} \phi \\ \psi \end{pmatrix}$ is an eigenvector of $\mathcal{A}_0^{0,\omega}$ with rank m associated with characteristic value ω_0, *i.e.*, there exist ϕ^ω and ψ^ω, holomorphic as functions of ω, such that $\phi^{\omega_0} = \phi$, $\psi^{\omega_0} = \psi$, and

$$
\mathcal{A}_0^{0,\omega} \begin{pmatrix} \phi^\omega \\ \psi^\omega \end{pmatrix} = (\omega - \omega_0)^m \begin{pmatrix} \widetilde{\phi}^\omega \\ \widetilde{\psi}^\omega \end{pmatrix},
$$

for some $\begin{pmatrix} \widetilde{\phi}^\omega \\ \widetilde{\psi}^\omega \end{pmatrix} \in L^2(\partial D)^2$. In other words, the following identities hold on ∂D:

$$
\tilde{\mathcal{S}}^\omega \phi^\omega - \frac{1}{\omega^2} \int_{\partial D} \psi^\omega d\sigma = (\omega - \omega_0)^m \widetilde{\phi}^\omega,
$$

$$
\left(\frac{1}{2}I - (\tilde{\mathcal{K}}^\omega)^* \right) \phi^\omega + \left(\frac{1}{2}I + (\mathcal{K}^{0,0})^* \right) \psi^\omega = (\omega - \omega_0)^m \widetilde{\psi}^\omega.
$$

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It then follows from (2.45) that

$$\tilde{\mathcal{S}}^\omega \phi^\omega - \frac{1}{|Y \setminus D|\omega^2} \int_{\partial D} \left(-\frac{1}{2}I + (\tilde{\mathcal{K}}^\omega)^* \right) \phi^\omega d\sigma$$

$$= \tilde{\mathcal{S}}^\omega \phi^\omega - \frac{1}{|Y \setminus D|\omega^2} \int_{\partial D} \left(\frac{1}{2}I + (\mathcal{K}^{0,0})^* \right) \psi^\omega d\sigma + \frac{(\omega - \omega_0)^m}{|Y \setminus D|\omega^2} \int_{\partial D} \tilde{\psi}^\omega d\sigma$$

$$= \tilde{\mathcal{S}}^\omega \phi^\omega - \frac{1}{\omega^2} \int_{\partial D} \psi^\omega d\sigma + \frac{(\omega - \omega_0)^m}{|Y \setminus D|\omega^2} \int_{\partial D} \tilde{\psi}^\omega d\sigma$$

$$= (\omega - \omega_0)^m \left(\tilde{\phi}^\omega + \frac{1}{|Y \setminus D|\omega^2} \int_{\partial D} \tilde{\psi}^\omega d\sigma \right).$$

Let

$$\eta^\omega := \left(\tilde{\phi}^\omega + \frac{1}{|Y \setminus D|\omega^2} \int_{\partial D} \tilde{\psi}^\omega d\sigma \right) \quad \text{and} \quad \mathbf{u}^\omega := \tilde{\mathcal{S}}^\omega \phi^\omega.$$

Then $\mathbf{u}^\omega$ satisfies

$$\begin{cases} (\mathcal{L}^{\tilde{\lambda},\tilde{\mu}} + \omega^2)\mathbf{u}^\omega = 0 & \text{in } D, \\ \mathbf{u}^\omega = \frac{1}{|Y \setminus D|\omega^2} \int_{\partial D} \frac{\partial \mathbf{u}^\omega}{\partial \tilde{\nu}} d\sigma + (\omega - \omega_0)^m \eta^\omega & \text{on } \partial D. \end{cases}$$

By Green's formula, we have

$$(\omega^2 - \omega_0^2) \int_D \mathbf{u}^\omega \cdot \overline{\mathbf{u}^{\omega_0}}$$

$$= \int_{\partial D} \mathbf{u}^\omega \cdot \frac{\partial \overline{\mathbf{u}^{\omega_0}}}{\partial \tilde{\nu}} - \overline{\mathbf{u}^{\omega_0}} \cdot \frac{\partial \mathbf{u}^\omega}{\partial \tilde{\nu}} d\sigma$$

$$= \left(\frac{1}{\omega^2} - \frac{1}{\omega_0^2} \right) \frac{1}{|Y \setminus D|} \int_{\partial D} \frac{\partial \mathbf{u}^\omega}{\partial \tilde{\nu}} d\sigma \cdot \int_{\partial D} \frac{\partial \overline{\mathbf{u}^{\omega_0}}}{\partial \tilde{\nu}} d\sigma + (\omega - \omega_0)^m \int_{\partial D} \eta^\omega \cdot \frac{\partial \overline{\mathbf{u}^{\omega_0}}}{\partial \tilde{\nu}} d\sigma.$$

Dividing by $\omega^2 - \omega_0^2$ and letting $\omega \to \omega_0$, we obtain

$$\int_D |\mathbf{u}^{\omega_0}|^2 + \frac{1}{2|Y \setminus D|\omega_0^4} \left| \int_{\partial D} \frac{\partial \mathbf{u}^{\omega_0}}{\partial \tilde{\nu}} d\sigma \right|^2 = \lim_{\omega \to \omega_0} \frac{(\omega - \omega_0)^m}{\omega^2 - \omega_0^2} \int_{\partial D} \eta^{\omega_0} \cdot \frac{\partial \overline{\mathbf{u}^{\omega_0}}}{\partial \tilde{\nu}} d\sigma.$$

Since the term on the left is nonzero, we conclude that $m = 1$. This completes the proof. $\square$

Analogously to Theorem 3.9, the following asymptotic formula for $\alpha = 0$ holds.

Theorem 3.12 *Suppose $\alpha = 0$. Let $\tilde{\omega}_0^2$ (with $\tilde{\omega}_0 > 0$) be a simple eigenvalue of (3.27). Then there exists a unique characteristic value $\omega_\mu^0 > 0$ of $\mathcal{A}^{0,\omega}$ lying in a small complex neighborhood V of $\tilde{\omega}_0$ and the following asymptotic expansion holds:*

$$\omega_\mu^0 - \tilde{\omega}_0 = \frac{1}{2\pi\sqrt{-1}} \sum_{p=1}^{+\infty} \frac{1}{p} \sum_{n=p}^{+\infty} \frac{1}{\mu^n} \operatorname{tr} \int_{\partial V} \mathcal{B}_{n,p}^\omega d\omega, \tag{3.31}$$

where

$$\mathcal{B}_{n,p}^\omega = (-1)^p \sum_{\substack{n_1 + \cdots + n_p = n \\ n_i \geq 1}} (\mathcal{A}_0^{0,\omega})^{-1} \mathcal{A}_{n_1}^{0,\omega} \cdots (\mathcal{A}_0^{0,\omega})^{-1} \mathcal{A}_{n_p}^{0,\omega}. \tag{3.32}$$

3.2.3 The case when $|\alpha|$ is of order of $1/\sqrt{\mu}$.

In this subsection we derive an asymptotic expansion which is valid for $|\alpha|$ of order $O(1/\sqrt{\mu})$, not just for fixed $\alpha \neq 0$ or $\alpha = 0$, as has been considered in the previous subsections and give the limiting behavior of ω_μ^α in this case.

Suppose that $|\alpha|^2\mu$ goes to $0 < \tau < +\infty$ and denote $\bar{\bar{\tau}} = (\tau_{ij})_{i,j=1,2}$, with $\tau_{ij} = \lim \alpha_i\alpha_j\mu$. Then, following the same arguments as those in the proof of Lemma 3.10, we can show that the following problem

$$
\begin{cases}
(\mathcal{L}^{\tilde{\lambda},\tilde{\mu}} + \omega^2)\mathbf{u} = 0 & \text{in } D, \\
\mathbf{u} + \dfrac{1}{|Y \setminus D|}\left[\dfrac{1}{1 - \frac{\tau}{\omega^2}}I + \dfrac{\bar{\bar{\tau}}}{(\omega^2 - 2\tau)(1 - \frac{\tau}{\omega^2})}\right]\displaystyle\int_D \mathbf{u} = 0 & \text{on } \partial D,
\end{cases}
\tag{3.33}
$$

has a nontrivial solution if and only if ω is a real characteristic value of the operator-valued function

$$
\mathcal{A}_\tau^\omega = \begin{pmatrix}
\widetilde{\mathcal{S}}^\omega & -\dfrac{1}{\omega^2 - \tau}\left[I + \dfrac{\bar{\bar{\tau}}}{\omega^2 - 2\tau}\right]\displaystyle\int_{\partial D} \cdot\, d\sigma \\
\dfrac{1}{2}I - (\widetilde{\mathcal{K}}^\omega)^* & \dfrac{1}{2}I + (\mathcal{K}^{0,0})^*
\end{pmatrix}.
$$

We can also show that any eigenvector of $\mathcal{A}_\tau^\omega$ has rank one. Then if we denote by $\widetilde{\omega}_\tau$ a simple eigenvalue of (3.33) then

$$
\omega_\mu^\alpha - \widetilde{\omega}_\tau = \frac{1}{2\pi\sqrt{-1}}\frac{1}{\mu}\int_{\partial V} (\mathcal{A}_\tau^\omega)^{-1}\mathcal{A}_1^{0,\omega}\, d\omega + O(1/\mu^2),
\tag{3.34}
$$

as $\mu \to +\infty$ and $|\alpha|^2\mu \to \tau$. Here $\mathcal{A}_1^{0,\omega}$ is defined by the same formula as in the previous subsection.

Not surprisingly, this asymptotic expansion tends continuously to (3.22) and (3.31) as τ goes to $+\infty$ or 0, respectively.

3.3 Derivation of the leading order terms

For $\alpha \neq 0$, let us write down explicitly the leading order term in the expansion of $\omega_\mu^\alpha - \omega_0$. We first observe that

$$
(\mathcal{A}_0^{\alpha,\omega})^{-1} = \begin{pmatrix}
(\widetilde{\mathcal{S}}^\omega)^{-1} & 0 \\
\left(\dfrac{1}{2}I + (\mathcal{K}^{\alpha,0})^*\right)^{-1}\left(\dfrac{1}{2}I - (\widetilde{\mathcal{K}}^\omega)^*\right)(\widetilde{\mathcal{S}}^\omega)^{-1} & \left(\dfrac{1}{2}I + (\mathcal{K}^{\alpha,0})^*\right)^{-1}
\end{pmatrix}.
\tag{3.35}
$$

Next, we prove the following Lemma.

Lemma 3.13 *Let $\mathbf{u}_0$ be an eigenvector associated to the simple eigenvalue ω_0^2 and let $\varphi := \frac{\partial \mathbf{u}_0}{\partial \tilde{\nu}}|_-$ on ∂D. Then we have, in a neighborhood of ω_0,*

$$
(\widetilde{\mathcal{S}}^\omega)^{-1} = \frac{1}{\omega - \omega_0}T + \mathcal{Q}^\omega
\tag{3.36}
$$

where Q^ω is operators in $\mathcal{L}(H^2(\partial D)^2, L^2(\partial D)^2)$ holomorphic in ω, and T is defined by

$$T(f) := -\frac{\langle f, \varphi \rangle \varphi}{2\omega_0 \displaystyle\int_D |\mathbf{u}_0|^2}, \tag{3.37}$$

where $\langle \ , \ \rangle$ is the inner product on $L^2(\partial D)^2$.

Proof. By Lemma 3.7, there are operators T and Q^ω in $\mathcal{L}(H^2(\partial D)^2, L^2(\partial D)^2)$ such that $(\widetilde{\mathcal{S}}_D^\omega)^{-1}$ takes the form

$$(\widetilde{\mathcal{S}}^\omega)^{-1} = \frac{1}{\omega - \omega_0} T + Q^\omega \tag{3.38}$$

where Q^ω is holomorphic in ω. Since

$$\mathrm{Id} = (\widetilde{\mathcal{S}}^\omega)(\widetilde{\mathcal{S}}^\omega)^{-1} = \frac{1}{\omega - \omega_0} \widetilde{\mathcal{S}}^\omega T + \widetilde{\mathcal{S}}^\omega Q^\omega, \tag{3.39}$$

by letting $\omega \to \omega_0$, we have

$$\widetilde{\mathcal{S}}_D^{\omega_0} T = 0. \tag{3.40}$$

Similarly, we can show that

$$T \widetilde{\mathcal{S}}_D^{\omega_0} = 0. \tag{3.41}$$

It then follows from (3.40) and (3.41) that $\mathrm{Im}\, A = \mathrm{Ker}\, \widetilde{\mathcal{S}}^{\omega_0} = \mathrm{span}\{\varphi\}$ and $\mathrm{Ker}\, A = \mathrm{Im}\, \widetilde{\mathcal{S}}^{\omega_0} = \mathrm{span}\{\varphi\}^\perp$. Here $\mathrm{span}\{\varphi\}$ denotes the vector space spanned by φ. Therefore

$$T = C\langle \cdot, \varphi \rangle \varphi \tag{3.42}$$

for some constant C.

By Green's formula, we have for $x \in D$

$$\widetilde{\mathcal{S}}^\omega(\varphi)(x) = \widetilde{\mathcal{S}}^\omega \left(\frac{\partial \mathbf{u}_0}{\partial \nu} \Big|_- \right)(x) - \widetilde{\mathcal{D}}^\omega (\mathbf{u}_0)(x)$$
$$= (\omega^2 - \omega_0^2) \int_D \widetilde{\Gamma}^\omega(x - y) \mathbf{u}_0(y) dy - \mathbf{u}_0(x). \tag{3.43}$$

In particular, we get

$$\widetilde{\mathcal{S}}^\omega(\varphi)(x) = (\omega^2 - \omega_0^2) \int_D \widetilde{\Gamma}^\omega(x - y) \mathbf{u}_0(y) dy, \quad x \in \partial D. \tag{3.44}$$

By expanding $\widetilde{\Gamma}^\omega(x - y)$ in ω, we now have

$$\widetilde{\mathcal{S}}^\omega(\varphi)(x) = 2\omega_0(\omega - \omega_0) \int_D \widetilde{\Gamma}^{\omega_0}(x - y) \mathbf{u}_0(y) dy + (\omega - \omega_0)^2 A^\omega, \tag{3.45}$$

for some function A^ω holomorphic in ω. Therefore, we have

$$(\widetilde{\mathcal{S}}^\omega)^{-1} \left(2\omega_0 \int_D \widetilde{\Gamma}^{\omega_0}(x - y) \mathbf{u}_0(y) dy \right) = \frac{1}{\omega - \omega_0} \varphi + B^\omega, \tag{3.46}$$

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where B^ω is holomorphic in ω, which together with (3.38) implies that

$$T\left(2\omega_0 \int_D \widetilde{\Gamma}^{\omega_0}(\cdot - y)\mathbf{u}_0(y)dy\right) = \varphi. \tag{3.47}$$

Note that if we take $\omega = \omega_0$ in (3.43), then

$$\mathbf{u}_0(x) = -\widetilde{\mathcal{S}}^{\omega_0}(\varphi)(x), \quad x \in D. \tag{3.48}$$

It then follows from (3.42) and (3.47) that

$$1 = C\left\langle 2\omega_0 \int_D \widetilde{\Gamma}^{\omega_0}(x - y)\mathbf{u}_0(y)dy, \varphi \right\rangle$$
$$= 2C\omega_0 \left\langle \mathbf{u}_0, \widetilde{\mathcal{S}}^\omega \varphi \right\rangle = -2C\omega_0 \int_D |\mathbf{u}_0|^2.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Because of (3.48), we have

$$(\frac{1}{2}I - (\widetilde{\mathcal{K}}^{\omega_0})^*)(\varphi) = \varphi \quad \text{on } \partial D. \tag{3.49}$$

Observe from (3.18) and (3.19) that the diagonal elements of $(\mathcal{A}_0^{\alpha,\omega})^{-1}\mathcal{A}_1^{\alpha,\omega}$ are 0 and

$$-(\frac{1}{2}I + (\mathcal{K}^{\alpha,0})^*)^{-1}(\frac{1}{2}I - (\widetilde{\mathcal{K}}^{\omega_0})^*)(\widetilde{\mathcal{S}}^\omega)^{-1}\mathcal{S}_1^{\alpha,\omega} + \text{an operator holomorphic in } \omega. \tag{3.50}$$

The identity (3.15) implies that $\mathcal{S}_1^{\alpha,\omega} = \mu\mathcal{S}^{\alpha,0}$, and hence it follows from (3.36) that

$$\frac{1}{2\pi\sqrt{-1}} \text{ tr} \int_{\partial V} (\mathcal{A}_0^{\alpha,\omega})^{-1}\mathcal{A}_1^{\alpha,\omega}d\omega = -\mu\text{tr}\left[T\mathcal{S}^{\alpha,0}(\frac{1}{2}I + (\mathcal{K}^{\alpha,0})^*)^{-1}(\frac{1}{2}I - (\widetilde{\mathcal{K}}^{\omega_0})^*)\right].$$

Since $\text{Im }T = \text{span}\{\varphi\}$, it follows from (3.49) that

$$\text{tr}\left[T\mathcal{S}^{\alpha,0}(\frac{1}{2}I + (\mathcal{K}^{\alpha,0})^*)^{-1}(\frac{1}{2}I - (\widetilde{\mathcal{K}}^{\omega_0})^*)\right]$$
$$= \frac{\left\langle \left[T\mathcal{S}^{\alpha,0}(\frac{1}{2}I + (\mathcal{K}^{\alpha,0})^*)^{-1}(\frac{1}{2}I - (\widetilde{\mathcal{K}}^{\omega_0})^*)\right]\varphi, \varphi \right\rangle}{\|\varphi\|_{L^2(\partial D)}^2}$$
$$= \frac{\left\langle \left[T\mathcal{S}^{\alpha,0}(\frac{1}{2}I + (\mathcal{K}^{\alpha,0})^*)^{-1}\right]\varphi, \varphi \right\rangle}{\|\varphi\|_{L^2(\partial D)}^2}. \tag{3.51}$$

We set

$$\mathbf{v}_0(x) := \mu\mathcal{S}^{\alpha,0}\left(\frac{1}{2}I + (\mathcal{K}^{\alpha,0})^*\right)^{-1}(\varphi)(x), \quad x \in Y \setminus \overline{D}. \tag{3.52}$$

Then $\mathbf{v}_0$ is the unique α-quasi-periodic solution to

$$\begin{cases} \mathcal{L}^{\lambda,\mu}\mathbf{v}_0 = 0 & \text{in } Y \setminus \overline{D}, \\ \dfrac{\partial\mathbf{v}_0}{\partial\nu}\Big|_+ = \mu\dfrac{\partial\mathbf{u}_0}{\partial\widetilde{\nu}}\Big|_- & \text{on } \partial D, \end{cases} \tag{3.53}$$

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and

$$\frac{1}{2\pi\sqrt{-1}} \operatorname{tr} \int_{\partial V} (\mathcal{A}_0^{\alpha,\omega})^{-1} \mathcal{A}_1^{\alpha,\omega} d\omega = \frac{1}{||\varphi||_{L^2}^2} \langle \varphi, T\mathbf{v}_0 \rangle$$

$$= \frac{1}{\mu} \frac{\displaystyle\int_{Y\setminus\overline{D}} \lambda|\nabla \cdot \mathbf{v}_0|^2 + \frac{\mu}{2}|\nabla\mathbf{v}_0 + \nabla\mathbf{v}_0^t|^2}{2\omega_0 \displaystyle\int_D |\mathbf{u}_0|^2}.$$

Thus the following corollary holds.

Corollary 3.14 *Suppose $\alpha \neq 0$. Then the following asymptotic formula holds:*

$$\omega_\mu^\alpha - \omega_0 = -\frac{1}{\mu} \frac{\displaystyle\int_{Y\setminus\overline{D}} \frac{\lambda}{\mu}|\nabla \cdot \mathbf{v}_0|^2 + \frac{1}{2}|\nabla\mathbf{v}_0 + \nabla\mathbf{v}_0^t|^2}{2\omega_0 \displaystyle\int_D |\mathbf{u}_0|^2} + O\left(\frac{1}{\mu^2}\right) \quad as \ \mu \to +\infty. \qquad (3.54)$$

When $\alpha = 0$, it does not seem to be likely that we can explicitly compute the leading order term in a closed as in the case $\alpha \neq 0$. But, let us now briefly explain how to compute the leading order term in the asymptotic expansion of $\omega_\mu^0 - \widetilde{\omega}_0$.

Let $\mathbf{u}_0$ be the (normalized) eigenvector of (3.27) associated with the simple eigenvalue $\widetilde{\omega}_0$. Let $(\widetilde{\phi}_0, \widetilde{\psi}_0)$ satisfy (3.30) with $\mathbf{u}$ replaced by $\mathbf{u}_0$ and $\omega = \widetilde{\omega}_0$. Since $\widetilde{\omega}_0$ is the only simple pole in V of the mapping $\omega \mapsto (\mathcal{A}_0^{0,\omega})^{-1}$, it can be proved that

$$(\mathcal{A}_0^{0,\omega})^{-1} = \frac{1}{\omega - \widetilde{\omega}_0} \left(\frac{d}{d\omega} \mathcal{A}_0^{0,\omega}\Big|_{\omega=\widetilde{\omega}_0} \begin{pmatrix} \widetilde{\phi}_0 \\ \widetilde{\psi}_0 \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\phi}_0 \\ \widetilde{\psi}_0 \end{pmatrix}\right)^{-1} \begin{pmatrix} \langle \cdot, \widetilde{\phi}_0 \rangle \widetilde{\phi}_0 & 0 \\ 0 & \langle \cdot, \widetilde{\psi}_0 \rangle \widetilde{\psi}_0 \end{pmatrix} \quad (3.55)$$

$$+ \text{ operator-valued function holomorphic in } \omega,$$

which allows us to explicit the leading-order term in the expansion of $\omega_\mu^0 - \widetilde{\omega}_0$. Similar calculations and expressions in the transition region ($|\alpha| = O(1/\sqrt{\mu})$) can be derived as well.

3.4 Criterion for gap opening

Following Hempel and Lienau [20], we provide in this subsection a criterion for gap opening in the spectrum of the operator given by (1.1) as $\mu \to +\infty$.

Let ω_j be the eigenvalues of $-\mathcal{L}^{\widetilde{\lambda},\widetilde{\mu}}$ in D with the Dirichlet boundary condition. Let $\widetilde{\omega}_j$ denote the eigenvalues of (3.27). We first prove the following min-max characterization of ω_j and $\widetilde{\omega}_j$.

Lemma 3.15 *The following min-max characterizations of ω_j^2 and $\widetilde{\omega}_j^2$ hold:*

$$\omega_j^2 = \min_{N_j} \max_{\mathbf{u} \in N_j, ||\mathbf{u}||=1} \widetilde{\mathbf{E}}(\mathbf{u}, \mathbf{u}), \qquad (3.56)$$

and

$$\widetilde{\omega}_j^2 = \min_{N_j} \max_{\mathbf{u} \in N_j, ||\mathbf{u}||=1} \frac{\widetilde{\mathbf{E}}(\mathbf{u}, \mathbf{u})}{1 - |\int_D \mathbf{u}|^2}, \qquad (3.57)$$

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where the minimum is taken over all j dimensional subspaces N_j of $(H_0^1(D))^2$. Here $H_0^1(D)$ is the set of all functions in $H^1(D)$ whose trace at ∂D is zero and $\widetilde{\mathbf{E}}$ is given by (2.24) with (λ, μ) replaced by $(\widetilde{\lambda}, \widetilde{\mu})$.

Proof. The identity (3.56) is well known. Note that if $\mathbf{v}$ satisfies the Dirichlet condition on ∂D, then $\widetilde{\mathbf{v}} := \mathbf{v} - \int_D \mathbf{v}$ satisfies the boundary condition

$$\widetilde{\mathbf{v}} + \frac{1}{|Y \setminus D|} \int_D \widetilde{\mathbf{v}} = 0 \quad \text{on } \partial D. \tag{3.58}$$

Conversely, if $\widetilde{\mathbf{v}}$ satisfies (3.58), then $\mathbf{v} := \widetilde{\mathbf{v}} + \frac{1}{|Y \setminus D|} \int_D \widetilde{\mathbf{v}}$ obviously satisfies the Dirichlet boundary condition.

Observe that the operator with the boundary condition in (3.27) is not self-adjoint, and hence Poincare's min-max principle can not be applied. So we now introduce an eigenvalue problem whose eigenvalues are exactly those of (3.27). Let $\mathcal{H} = \text{span}\{H_0^2(D), 1_Y\}$ in $L^2(Y)$ where $H_0^2(D)$ is regarded as a subspace of $L^2(Y)$ by extending the functions to be 0 in $Y \setminus D$. Let $\mathcal{G}$ be the closure of $\mathcal{H}$ in $L^2(Y)$. Define the operator $\mathbf{T} : \mathcal{H} \times \mathcal{H} \to \mathcal{G} \times \mathcal{G}$ by

$$\mathbf{Tu} = \begin{cases} -\mathcal{L}^{\widetilde{\lambda}, \widetilde{\mu}} \mathbf{u} & \text{on } D, \\ \frac{1}{|Y \setminus D|} \int_D \mathcal{L}^{\widetilde{\lambda}, \widetilde{\mu}} \mathbf{u} & \text{on } Y \setminus D. \end{cases}$$

The constant value of $\mathbf{Tu}$ in $Y \setminus D$ was chosen so that $\int_Y \mathbf{Tu} = 0$. Then one can easily see that $\mathbf{T}$ is a densely defined self-adjoint operator on $\mathcal{H} \times \mathcal{H}$ and

$$\langle \mathbf{Tu}, \mathbf{v} \rangle_Y = \widetilde{\mathbf{E}}(\mathbf{u}, \mathbf{v}) \quad \text{for } \mathbf{u}, \mathbf{v} \in \mathcal{H} \times \mathcal{H}. \tag{3.59}$$

One can also show that nonzero eigenvalues of $\mathbf{T}$ are eigenvalues of (3.27), and vice versa.

Let M_j be a j-dimensional subspace of $\mathcal{H} \times \mathcal{H}$ perpendicular to constant vectors which is eigenvectors corresponding to eigenvalue zero. Then by Poincare's min-max principle we have

$$\begin{aligned} \widetilde{\omega}_j^2 &= \min_{M_j} \max_{\mathbf{u} \in M_j} \frac{\langle \mathbf{Tu}, \mathbf{u} \rangle_Y}{\langle \mathbf{u}, \mathbf{u} \rangle_Y} \\ &= \min_{M_j} \max_{\mathbf{u} \in M_j} \frac{\widetilde{\mathbf{E}}(\mathbf{u}, \mathbf{u})}{\langle \mathbf{u}, \mathbf{u} \rangle_Y} \\ &= \min_{N_j} \max_{\mathbf{v} \in N_j} \frac{\widetilde{\mathbf{E}}(\mathbf{v} - \int_D \mathbf{v}, \mathbf{v} - \int_D \mathbf{v})}{\langle \mathbf{v} - \int_D \mathbf{v}, \mathbf{v} - \int_D \mathbf{v} \rangle_Y} \\ &= \min_{N_j} \max_{\mathbf{v} \in N_j} \frac{\widetilde{\mathbf{E}}(\mathbf{v}, \mathbf{v})}{\langle \mathbf{v}, \mathbf{v} \rangle_D - |\int_D \mathbf{v}|^2}. \end{aligned}$$

This complets the proof of the lemma. $\qquad \square$

Lemma 3.16 *The eigenvalues ω_j and $\widetilde{\omega}_j$ interlace in the following way:*

$$\omega_j \leq \widetilde{\omega}_j \leq \omega_{j+2}, \quad j = 1, 2, \ldots. \tag{3.60}$$

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Proof. Lemma 3.15 shows that the first inequality in (3.60) is trivial and we only have to prove the second one. Let $\mathbf{u}_j$ denotes the normalized eigenvectors associated with ω_j. Let N_{j+2} denotes the span of the eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_{j+2}$ and $\widetilde{N}$ be the subspace of N_{j+2} composed of all the elements in N_{j+2} which have zero integral over D. Since the set of constant vectors has dimension two, $\widetilde{N}$ is of dimension greater than j. Therefore, we have $\widetilde{\omega}_j \leq \omega_{j+2}$, as desired. $\qquad\square$

Since 0 is eigenvalue of the periodic problem with multiplicity 2, formulae (3.22), (3.31), and (3.34) show that the spectral bands converge, as $\mu \to \infty$, to

$$[0, \omega_1] \bigcup [0, \omega_2] \bigcup_{j \geq 1} [\widetilde{\omega}_j, \omega_{j+2}], \tag{3.61}$$

and hence we have a band-gap if and only if the following holds:

$$\omega_{j+1} < \widetilde{\omega}_j \quad \text{for some } j \quad (\textbf{Criterion for gap opening}). \tag{3.62}$$

Observe that by (3.56) and (3.57) the gap opening criterion is equivalent to

$$\min_{N_{j+1}} \max_{\mathbf{u} \in N_{j+1}, ||\mathbf{u}||=1} \widetilde{\mathbf{E}}(\mathbf{u}, \mathbf{u}) < \min_{N_j} \max_{\mathbf{u} \in N_j, ||\mathbf{u}||=1} \frac{\widetilde{\mathbf{E}}(\mathbf{u}, \mathbf{u})}{1 - |\int_D \mathbf{u}|^2}, \tag{3.63}$$

where N_j is a j-dimensional subspace of $H_0^1(\partial D)^2$.

It is not likely to find conditions on the inclusion D so that the gap-opening criterion is satisfied by rigorous analysis. However, finding such conditions by all means such as numerical computations will be of great importance. It should be emphasized that the criterion (3.62) is for the case when the matrix and the inclusion have the same density, which we assumed to be equal to 1.

4   Gap opening criterion when densities are different

We now consider periodic elastic composites such that the matrix and the inclusion have different densities.

Suppose that the density of the matrix is ρ while that of the inclusion is 1 (after normalization). The Lamé parameters are the same as before. In this case, the first equation of the eigenvalue problem (3.1) is changed to

$$\mathcal{L}^{\lambda,\mu}\mathbf{u} + \rho\omega^2\mathbf{u} = 0, \quad \text{in } Y \setminus \overline{D}. \tag{4.1}$$

Hence we can show by exactly the analysis that the asymptotic expansions (3.22), (3.31), and (3.34) hold if we replace the operators (3.25) and (3.19) (and (3.26)) with new operators (depending on the density ρ)

$$\mathcal{A}_0^{0,\omega} = \begin{pmatrix} \tilde{\mathcal{S}}^\omega & -\dfrac{1}{\rho\omega^2}\displaystyle\int_{\partial D} \cdot\, d\sigma \\[2mm] \dfrac{1}{2}I - (\tilde{\mathcal{K}}^\omega)^* & \dfrac{1}{2}I + (\mathcal{K}^{0,0})^* \end{pmatrix} \tag{4.2}$$

and

$$\mathcal{A}_l^{\alpha,\omega} = \rho^{l-1} \begin{pmatrix} 0 & -\mathcal{S}_l^{\alpha,\omega} \\ 0 & \frac{\rho}{\mu}(\mathcal{K}_{l+1}^{\alpha,\omega})^* \end{pmatrix}, \quad l \geq 1, \tag{4.3}$$

respectively, and the eigenvalue problems (3.27) and (3.33) with the following eigenvalue problems

$$\begin{cases} (\mathcal{L}^{\tilde{\lambda},\tilde{\mu}} + \omega^2)\mathbf{u} = 0 & \text{in } D, \\ \mathbf{u} + \dfrac{1}{\rho|Y \setminus D|} \displaystyle\int_D \mathbf{u} = 0 & \text{on } \partial D. \end{cases} \tag{4.4}$$

and

$$\begin{cases} (\mathcal{L}^{\tilde{\lambda},\tilde{\mu}} + \omega^2)\mathbf{u} = 0 & \text{in } D, \\ \mathbf{u} + \dfrac{1}{\rho|Y \setminus D|}\left[\dfrac{1}{1-\frac{\tau}{\rho\omega^2}}I + \dfrac{\bar{\bar{\tau}}}{(\rho\omega^2 - 2\tau)(1-\frac{\tau}{\rho\omega^2})}\right]\displaystyle\int_D \mathbf{u} = 0 & \text{on } \partial D, \end{cases} \tag{4.5}$$

respectively.

Let $\{\widetilde{\omega}_j\}$ be the set of eigenvalues of (4.4). In order to express $\widetilde{\omega}_j$ using the min-max principle, we define $\langle \ , \ \rangle_Y$ by

$$\langle \mathbf{u}, \mathbf{v} \rangle_Y = \int_D \mathbf{u} \cdot \mathbf{v} + \rho \int_{Y \setminus D} \mathbf{u} \cdot \mathbf{v}. \tag{4.6}$$

We also define, as before, $\mathbf{T}$ to be

$$\mathbf{Tu} = \begin{cases} -\mathcal{L}^{\tilde{\lambda},\tilde{\mu}}\mathbf{u} & \text{on } D, \\ \dfrac{1}{\rho|Y \setminus D|} \displaystyle\int_D \mathcal{L}^{\tilde{\lambda},\tilde{\mu}}\mathbf{u} & \text{on } Y \setminus D. \end{cases} \tag{4.7}$$

Then $\mathbf{T}$ is self-adjoint with respect to $\langle \ , \ \rangle_Y$. By Poincare's min-max principle again, we have

$$\begin{aligned} \widetilde{\omega}_j^2 &= \min_{M_j} \max_{\mathbf{u} \in M_j} \frac{\langle \mathbf{Tu}, \mathbf{u} \rangle_Y}{\langle \mathbf{u}, \mathbf{u} \rangle_Y} \\ &= \min_{M_j} \max_{\mathbf{u} \in M_j} \frac{\widetilde{\mathbf{E}}(\mathbf{u}, \mathbf{u})}{\langle \mathbf{u}, \mathbf{u} \rangle_Y} \\ &= \min_{N_j} \max_{\mathbf{v} \in N_j} \frac{\widetilde{\mathbf{E}}(\mathbf{v}, \mathbf{v})}{\left\langle \mathbf{v} - \frac{1}{|D|+\rho|Y \setminus D|}\int_D \mathbf{v}, \mathbf{v} - \frac{1}{|D|+\rho|Y \setminus D|}\int_D \mathbf{v} \right\rangle_Y} \\ &= \min_{N_j} \max_{\mathbf{v} \in N_j} \frac{\widetilde{\mathbf{E}}(\mathbf{v}, \mathbf{v})}{\langle \mathbf{v}, \mathbf{v} \rangle_D - \frac{1}{|D|+\rho|Y \setminus D|}\left|\int_D \mathbf{v}\right|^2}, \end{aligned}$$

where M_j and N_j are the same as in the proof of Lemma 3.15. Therefore, we have the following min-max characterization of the eigenvalues of the problem (4.4):

$$\widetilde{\omega}_j^2 = \min_{N_j} \max_{\mathbf{u} \in N_j} \frac{\widetilde{\mathbf{E}}(\mathbf{u}, \mathbf{u})}{\langle \mathbf{u}, \mathbf{u} \rangle_D - \frac{1}{|D|+\rho|Y \setminus D|}\left|\int_D \mathbf{u}\right|^2}. \tag{4.8}$$

We then get a band-gap criterion for the different density case which is equivalent to (3.62):

$$\min_{N_{j+1}} \max_{\mathbf{u} \in N_{j+1}, ||\mathbf{u}||=1} \widetilde{\mathbf{E}}(\mathbf{u}, \mathbf{u}) < \min_{N_j} \max_{\mathbf{u} \in N_j, ||\mathbf{u}||=1} \frac{\widetilde{\mathbf{E}}(\mathbf{u}, \mathbf{u})}{1 - \frac{1}{|D| + \rho|Y \setminus D|} \left| \int_D \mathbf{u} \right|^2}. \tag{4.9}$$

It is quite interesting to compare (4.9) with (3.63). If $\rho < 1$, then

$$\min_{N_j} \max_{\mathbf{u} \in N_j, ||\mathbf{u}||=1} \frac{\widetilde{\mathbf{E}}(\mathbf{u}, \mathbf{u})}{1 - |\int_D \mathbf{u}|^2} < \min_{N_j} \max_{\mathbf{u} \in N_j, ||\mathbf{u}||=1} \frac{\widetilde{\mathbf{E}}(\mathbf{u}, \mathbf{u})}{1 - \frac{1}{|D| + \rho|Y \setminus D|} \left| \int_D \mathbf{u} \right|^2}, \tag{4.10}$$

which shows that smaller the density ρ is, wider the band-gap is, provided that (3.62) is fulfilled. This phenomenon was reported by Economou and Sigalas [14] who observed that periodic elastic composites whose matrix has lower density and higher shear modulus compared to those of inclusions yield better open gaps, and the analysis of this paper agrees with it.

5   Conclusion

In this paper, we have reduced band structure calculations for phononic crystals to the problem of finding the characteristic values of a family of meromorphic integral operators. We have also provided complete asymptotic expansions of these characteristic values as the Lamé parameter μ goes to infinity, established a connection between the band gap structure and the Dirichlet eigenvalue problem for the Lamé operator, and given a criterion for gap opening as μ becomes large. The leading-order terms in the expansions of the characteristic values are explicitly computed. An asymptotic analysis for the band-gap structure in three-dimensions can be provided with only minor modifications of the techniques presented here. Our results in this paper open the road to numerous numerical and analytical investigations on phononic crystals and could, in particular, be used for systematic optimal design of phononic structures.

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